1. Vector product

$$\bar{A} \times \bar{B} = AB \sin\theta \ \bar{C} \ (\text{vector})$$

where $\bar{C}$ is $\perp$ to A & B and [illegible]

2. Scalar [illegible]

$$\bar{A} \cdot \bar{B} = AB \cos\theta \ (\text{scalar})$$

ANALYTIC MECHANICS

BY

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PREFACE

We have attempted to write a rigorous, teachable introduction to the study of mechanics. We believe that certain fundamental principles of mechanics, used in common by students in the various branches of engineering, in theoretical physics, and in celestial mechanics are essential to the satisfactory progress of a student in any of those fields. To this end, we have chosen as our subject matter only such fundamental theorems. We have not attempted to develop the special methods that are used in some branches of these sciences but not in others, because having fixed the principles on which these methods are based, they are better studied in treatises on those subjects. We have, however, indicated the fields in which certain processes are employed and have told the student where more detailed information can be found.

One purpose of the work is to develop facility in the application of mathematical formulæ to the investigation of physical phenomena. We have observed about the same degree of rigor that is observed in the best modern undergraduate texts in mathematics. We have been careful to state the assumptions that have been made and the conditions which must prevail before a given mathematical process is applicable, and have then proved rigorously the theorems based on the stated conditions.

We have quoted freely the theorems that students who have completed a first course in the calculus are supposed to know. In this sense the book is mathematical.

The illustrative problems that are solved and the exercises which are set for the student are chosen largely from engi-

neering fields. They are real problems chosen from real structures or real machines. In this sense the book is a mechanics of engineering.

We have chosen the older methods of treatment, such as the resolution of forces, etc., because we believe that in this way the student keeps in a little closer touch, at least in the beginning, with the physics of the problem.

While we have tried always to be rigorous, our ambition has been to produce a book that is distinctively teachable. We have made many "remarks" to students, in which we have given directions and cautions in the applications of the fundamental theorems to practical problems. These "remarks" are not mere directions for the solutions of problems, but fundamental discussions of the strength and limitations of basic methods which, once mastered, will be of service throughout the student's professional career.

The book may be adapted to a number of possible courses, emphasizing more or less the engineering or the mathematical phase of the subject, as the reader desires. He may omit, without breaking the logical sequence, any article in fine print, if he omits all succeeding articles in fine print. He may, to reduce the amount of mathematics necessary, omit articles 95–106, from Center of Gravity, and articles 229–242, from Moment of Inertia. If he desires to eliminate parts applicable especially to the analysis of structures, he may omit articles 87–88 (three-force-pieces), and may abbreviate further by omitting articles 85–88. He may omit either all of Chapter iv, or articles 122–124, or all of Chapter viii, or all of Chapter xiii, or any number of these, so far as requisition made on them by succeeding articles is concerned.

We have used $\frac{W}{g}$ instead of "mass."

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SWARTHMORE COLLEGE,
June, 1915.

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ANALYTIC MECHANICS

INTRODUCTORY

1. Mechanics is the *science of motion*. It is a natural science in the sense that the laws of motion have been established by observation and experimentation. These laws and principles being assumed, it is the province of Theoretical Mechanics to deduce consequences from them; to predict, being given certain conditions, the motions and positions of bodies.

The thing that moves is *matter*. A *body* is a definite finite portion of matter. A *particle* is a body whose length, breadth, and thickness is infinitesimally small. The limit which such a body approaches as its dimensions are decreased is a geometrical point endowed with matter. The position of this point is said to be the position of the particle.

The locus of all the positions occupied by a particle as it moves from a position A to a position B is called its *path*. The length of its path is called the *space* through which it moves.

Since no particle can occupy two positions at the same instant, we know that some *time* elapsed while the particle was moving from A to B.

2. From the foregoing we should suspect that a science of motion involves at least three fundamental and distinct elements, viz.: matter, space, and time. In fact it can be proved that all physical quantities can be expressed in terms of three distinct elements, and experience has shown that the expression is simpler in terms of matter, space, and time, than in any other. We shall define many other terms, such as force, velocity, momentum, and so on, but these, by the aid of certain axioms stated later, will be expressed in terms of the fundamental elements.

3. Mass is *measured matter.* That is, the mass of a body is a number expressing the ratio of the quantity of matter in that body to the quantity of matter in another body, chosen arbitrarily, as a unit. The unit of mass is a gram, which is defined as $\frac{1}{1000}$ of a piece of platinum-iridium kept in the laboratory of the International Committee of Weights and Measures at Sevres, France; or, an avoirdupois pound, which in the United States is defined as $\dfrac{1}{2.20462}$ of a kilogram. Very exact copies of the units of mass exist in many places. The weights for balances used in our laboratories and in commerce are more or less exact copies of those units, or multiples of those units.

4. Length is *measured extension.* That is, the length of a line is the *number* of times that it contains a given unit. The unit of length used most frequently in mechanics is the centimeter, or the foot. The centimeter is $\frac{1}{100}$ of the distance between two marks on a bar of platinum-iridium, at a temperature of 0° Centigrade, and a pressure of 760 millimeters of mercury. This bar is preserved in the laboratory of the International Committee of Weights and Measures. The foot is one-third of a yard, which in the United States is defined as $\frac{3600}{3937}$ of a meter.

The meter, foot, and yard, used in commerce, are approximate reproductions of the units of length.

5. Time is *measured duration.* The unit of time usually is a mean solar day or a mean solar minute or a mean solar second. A mean solar day is the time that elapses between two successive passages of the mean sun over the meridian. An hour is one twenty-fourth of a day. A minute is one sixtieth of an hour; a second, one-sixtieth of a minute. Thus mass, length, and time are expressed in numbers to which we may apply the laws of ordinary algebra. We shall adopt as units of length, mass, and time, one foot, one pound, and one mean solar second, respectively.

6. *All motion is relative.* When we say that a body is at rest, we mean, simply, rest relative to another body. For example, a man riding in a moving carriage is at rest relative to the carriage, but is in motion relative to the ground. A book lying on a table is at rest with respect to the table, but with respect to the sun or the stars, it is moving. In the problems that we shall consider we shall refer our motions to objects in the customary way. We shall say a building is at rest, meaning always with regard to the earth. No confusion is likely to occur. It is only necessary at any time to fix definitely in mind the body or bodies with respect to which we wish to define the motion.

In the work that follows, the position of a particle usually will be referred to an origin and two lines — the x-axis and the y-axis — through it, just as in Analytic Geometry. In this course we rarely need to consider motion except with reference to these lines.

7. To give the path (Art. 1) by which a particle moves from A to B will not completely describe its motion; for two particles may describe the same path at different speeds. We must, therefore, give also its *velocity*, that is its rate of change of position.

8. Velocity may be either uniform or variable. If a particle moves in a straight line and moves over equal spaces in equal times, its velocity is said to be uniform. A velocity which is not uniform is variable.

9. Mathematical Expression for Velocity. — If a particle is moving with uniform velocity, and describes a space s in a time t, then if v be the velocity, we have from definition

$$v = \frac{s}{t}. \qquad \qquad (1)$$

If the particle is moving with a variable velocity, and if it describes a space Δs in a time Δt, we define its velocity at any point as

$$v = \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t}.$$

Now Δs approaches zero as Δt approaches zero. We may therefore write the last equation:

$$v = \lim_{\Delta t \doteq 0}\left(\frac{\Delta s}{\Delta t}\right) = \frac{ds}{dt}. \quad . \quad . \quad . \quad . \quad (2)$$

10. Acceleration is the *rate of change of velocity.* Acceleration may be either constant or variable.

If $\Delta v =$ the change of the velocity of a particle in time Δt, we define the acceleration a by the equation:

$$a = \lim_{\Delta t \doteq 0}\left(\frac{\Delta v}{\Delta t}\right)$$

$$= \frac{dv}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

from (2) $$\frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

$$\therefore\ a = \frac{d^2s}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

11. That which moves matter is *force.* The relations between mass, space, time, and force have been expressed by Sir Isaac Newton in what are known as Newton's Laws of Motion.

LAW I. *Every body persists in a state of rest or of uniform motion in a straight line except in so far as it may be compelled by force to change the state.*

LAW II. *Change of motion is proportionate to the force applied and takes place in the direction of the straight line in which the force acts.*

LAW III. *To every action there is an equal and opposite reaction, or, the mutual actions of any two bodies are always equal and oppositely directed.*

Strictly speaking, these Laws of Motion apply only to bodies that are not rotating. Law I is the criterion for the existence of a force. If we observe a body that is not at rest, or is not moving uniformly in a straight line, we say that there

is a force acting upon it. Having found that force is acting on a body, we next desire to measure the force and to find its direction. We are enabled to do this by the second Law.

12. Momentum is the *mass of a moving body multiplied by its velocity.* Change of motion in the second law means the rate of change of momentum.

If $F =$ Force acting on a body;

$M =$ Mass of the body;

$v =$ Velocity of the body;

then the first part of the Law II may be written

$$F = k\frac{d}{dt}(Mv),$$

where k is a factor which depends on the units employed. In almost every instance M is constant. Hence in most cases we may write

$$F = kM\frac{dv}{dt} = kMa \text{ (by equation (3))} \quad . \quad . \quad . \quad (5)$$

If in addition our units are so chosen that $F = 1$, $M = 1$, when $a = 1$, then $k = 1$ and our equation becomes

$$F = Ma.$$

That is, Force = Mass times Acceleration. This is one of the most important equations in mechanics.

The second part of Law II tells us the direction in which the force acts.

PART I. STATICS

CHAPTER I

THE COMPOSITION AND RESOLUTION OF FORCES ACTING ON A PARTICLE

13. Force is that which changes, or tends to change, the motion of matter. It is either a push or a pull. The idea of force is almost intuitional. In a general way we have known that if a body is moving it is acted upon by some force, and that the greater the body the greater the force required to give it a certain velocity; that is, that there are forces of different magnitudes. One can convince himself by a series of experiments that a body, a book lying on a table for instance, will behave in different ways, if it be acted on by forces of different magnitudes; or, by forces having different lines of action, or, if the point at which a force is applied to the body is changed. Accordingly we say that a force is completely determined if we know its

1. Point of application.

2. Line of action and sense (that is, direction along the line of action).

3. Magnitude.

14. Measurement of Force. — The weight of a unit mass, determined by a spring balance, may be taken as a unit force. There is this objection to this practice, — the weight of a mass varies with its latitude and its distance from the center of the earth. A mass weighs less at the equator than at the poles of the earth, and less on a mountain top than at sea level. This variation is very small,* and the error introduced by its

* A body weighing one pound at sea level, latitude 45°, weighs .99337 pound at sea level at the equator, and its weight decreases .000006 of a pound for each 100 feet elevation above sea level.

adoption is smaller than that introduced in assuming the homogeneity of materials of which we make our structures; hence, it is used almost exclusively in engineering practice, and, in fact, in all cases involving mechanical considerations, except in experiments requiring the utmost refinement. However, in order that we have a constant unit, we may make the following definition:

The unit of force is the weight of a unit mass at sea level at a latitude of 45° * (Art. 3). Knowing the weight of a body at any place, we can, by methods to be explained, compute its weight for any other place.

15. Representation of Forces. — We have shown in Art. 13, that a force is uniquely determined when its point of application, its line of action and sense, and its magnitude are known. Therefore, a force may be represented by the segment of a straight line.

For, let P be a force. If from O, its point of application, we draw a line in the direction in which the force is acting, lay off on this line OA a segment equal to as many units in length as the number which represents the magnitude of P, and affix the arrow head to denote the sense of P, we have represented the force P, completely. Also it is represented uniquely; that is, any one force may be represented by one and only one line; and a given segment, marked with an arrow head, can represent one and only one force.

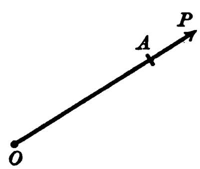


FIG. 1

In all cases, when we say a force is represented by a line, we mean represented completely. In some of the later theorems we shall have occasion to represent forces in some particular, but not to represent them completely. For example, we may represent the force in magnitude and direction, but

* At sea level at a latitude of 45° g (see Art. 142) has the value 980.665 centimeters per second. The unit force might be defined as the weight of a unit mass at any place where g has this standard value, and this is the unit used by the United States Bureau of Standards.

not in line of action. In all cases in which the force is not represented completely we shall specify in what particulars the representation is made.

16. A particle which is acted upon by an unbalanced system of forces will move in a certain direction and with a certain increase of velocity (Newton's second law). There is, however, a single force which, acting upon the particle, would produce the same effect. This force is called the resultant of the system. That is, the *resultant* of any number of forces acting on a particle is the single force which acting alone will produce the same effect as the given system of forces. We shall consider first two forces, quoting the fundamental and very important theorem known as the parallelogram of forces.

17. Parallelogram of Forces. — *If two forces acting on a particle are represented by two sides of a parallelogram, their resultant is represented by that diagonal of the parallelogram which passes through the intersection of these sides.*

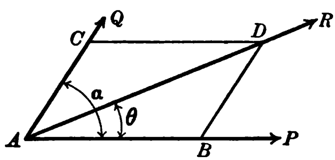


FIG. 2

That is, if two forces, P and Q, acting upon a particle at A are represented by AB and AC, their resultant R will be represented by the line AD. It will be noticed that the representation of the forces is complete; that is, they are represented in every particular.

18. The parallelogram law is a very important one. A great many proofs have been given for it, but the principles assumed in these proofs are no simpier than the theorem itself.* We shall assume the truth of this theorem just as we assumed the truth of Newton's Laws of Motion.

* A geometrical proof may be found in Todhunter's *Mechanics for Beginners*, p. 19 et seq.; or in *Traité de Mécanique* by Poisson, p. 45 et seq.; or in Loney's *Elements of Statics and Dynamics*, p. 25.

19. The forces which have a given force for their resultant are called the *components* of this force. The process of finding the resultant of any number of forces is known as the *composition* of forces. The process of finding the components of a given force is called the *resolution* of forces. Two systems of forces acting on a particle and having the same resultant are said to be *equivalent.*

In later articles we shall show that we can replace any system of forces acting on a *rigid body* by a single force, without changing the state of rest or motion of the body, except in one case which we shall point out later; and in this case, any system of forces can be replaced by two forces. Any two systems of forces that can be replaced by the same two forces are said to be *equivalent.* When it is possible to replace them by a *single* force we shall call this force their *resultant.*

20. The theorem of Art. 17 enables us to find the resultant of two given forces acting on a particle. Let AB and AC represent the forces P and Q respectively. Let α be the angle between them. Then

$$\overline{AD}^2 = \overline{AB}^2 + \overline{AC}^2 + 2\,\overline{AB}\cdot\overline{AC}\cos\alpha.$$

If, as is usual, we denote the magnitude of P, Q, and R by P, Q, and R, respectively, we shall have

$$R^2 = P^2 + Q^2 + 2\,PQ\cos\alpha. \quad . \quad . \quad . \quad . \quad (6)$$

Equation (6) gives the magnitude of the resultant. It will be noted that BD, being parallel and equal to AC represents in magnitude and direction (but not line of action) the force Q. Moreover,

$$\frac{AD}{\sin\alpha} = \frac{BD}{\sin\theta};$$

or

$$\frac{R}{\sin\alpha} = \frac{Q}{\sin\theta}.$$

This equation determines the direction of R with respect to P. Hence, R is completely determined.

21. Suppose that any number of forces P_1, P_2, P_3, . . . act on a particle. By the parallelogram law, one could find the resultant of two of them, and then the resultant of this first resultant, and one of the remaining forces. By repeating this process, it is evident that we could find the resultant of any number of forces acting on a particle.

22. The Polygon of Forces. — In Art. 20 we showed that the force Q may be represented in magnitude, direction, and sense, but not in line of action, by the line, BD (Fig. 2). We might, therefore, find the resultant of P and Q thus: Draw AB to represent P. At the extremity B draw a line parallel to Q and on this line lay off BD equal in length to the magnitude of Q. Draw AD. This evidently represents the resultant. This process can be easily extended to any number of forces. For example:

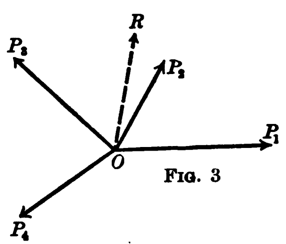


FIG. 3

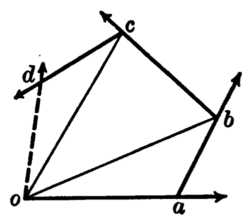


FIG. 4

Let a particle at O be acted upon by any number of forces, P_1, P_2, . . . , to be definite, say P_1, P_2, P_3, P_4. Let oa, ab, bc, cd, Fig. 4, be equal and parallel respectively to P_1, P_2, P_3, P_4; then ob is equal and parallel to the resultant of P_1 and P_2; oc is equal and parallel to the resultant of P_1, P_2, and P_3; the line od is equal and parallel to the resultant of all the forces. That is, *From any point O chosen arbitrarily to represent the point of application of the forces, draw a line oa parallel and equal to P_1; from the extremity a draw ab parallel and equal to P_2, and so on until all the forces are represented in magnitude and direction. A line drawn from o closing the polygon represents in*

magnitude and direction the resultant of all the forces. If o coincides with O, the closing side represents the resultant completely. This theorem, known as the Polygon of Forces, is at the foundation of Graphical Mechanics, a process used extensively by designers of roofs and bridges. The methods of Graphical Mechanics are powerful and elegant.*

23. REMARK. — It will be noted that the composition of two forces (Art. 19) is a unique process; that is, given two forces, P and Q, there is one and only one resultant. The process of resolution of forces (Art. 19) is, however, not unique, since with a given line segment as a diagonal an infinite number of parallelograms may be constructed; and hence, any force may be replaced by two forces in an infinite number of ways. It is usual to render the process definite by giving two additional conditions; such, for example, as the direction of the components, or the direction and magnitude of one of the components. The student may prove that when these conditions are introduced the problem becomes unique.

24. The Rectangular Components of a Force. — It is usual, when possible, to resolve a force into components at right angles to each other, called *rectangular components.* For example, if OX and OY are two lines at right angles to each other, it is evident from the parallelogram of forces that the component of R along OX equals $R \cos \theta$. That is, the *rectangular component of a force along any line equals the force times the cosine of the angle between the direction of the line and the force.* This rectangular component of a force along a line is often called the *resolved part* of the force along the line. The resolved part of a force along a line represents the entire effect of the force along the line.

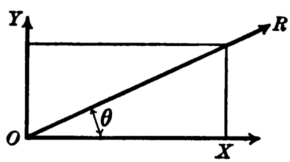


FIG. 5

25. Forces acting in one plane are called *coplanar* forces. Those acting along one line are said to be *collinear.* Those

* The reader who is interested may consult any one of numerous books, among them being *Roofs and Bridges*, Vol. II, by Merriman and Jacoby. It is not our purpose to develop these methods.

acting at a point are *concurrent*. Forces acting on a particle are therefore concurrent.

26. The Resultant of Any Number of Coplanar Forces acting on a Particle. The Analytic Expression. — Let any number of forces, P_1, P_2, ... P_n, act on a particle at O. To fix the ideas, suppose there are four forces, though the method is evidently general. Choose O as origin, OX and OY as axes of coördinates.

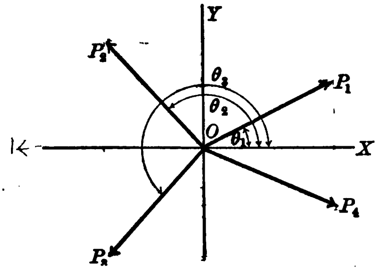


FIG 6

Let $\theta_1 =$ the angle between the positive direction of the x-axis and the positive direction of P_1.

Let $\theta_2 =$ the angle between the positive direction of the x-axis and the positive direction of P_2.

$\theta_i =$ the angle between the positive direction of the x-axis and the positive direction of P_i. ($i = 1, 2, \ldots n$.)

Replace P_1 by its rectangular components.

$P_1 \cos\theta_1 = X_1$ (say), along the x-axis, and $P_1 \sin\theta_1 = Y_1$ (say), along the y-axis, and in general, replace any force, P_i by its two rectangular components, $P_i \cos\theta_i = X_i$, along the x-axis and $P_i \sin\theta_i = Y_i$, along the y-axis.

Then our system of forces has been reduced to an equivalent system, Art. 19, each force of which acts along one or the other of the axes.

Let $X = \Sigma X_i =$ the sum of all the components along the x-axis

$$= P_1 \cos\theta_1 + P_2 \cos\theta_2 + \cdots \qquad = \Sigma P_i \cos\theta_i.$$

Let $Y = \Sigma Y_i =$ the sum of all the components along the y-axis

$$= P_1 \sin\theta_1 + P_2 \sin\theta_2 + \cdots \qquad = \Sigma P_i \sin\theta_i.$$

We have now replaced our system of forces by two forces, X and Y, at right angles to each other.

Let R be the resultant of the two forces X and Y, and therefore of the system of forces, $P_1, P_2, \cdots P_n$; then

$$R^2 = X^2 + Y^2$$

and $$R = \sqrt{X^2 + Y^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

We shall always choose the positive sign of the radical. This gives the *magnitude* of the resultant.

To find the *direction* of R.

Let $\theta =$ the angle between the positive direction of R and the positive direction of the x-axis.

$$\text{Then } \tan\theta = \frac{Y}{X}.$$

To determine the quadrant of θ we have either of the two additional equations,

$$\sin\theta = \frac{Y}{R}, \text{ or } \cos\theta = \frac{X}{R}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

27. Remark.—It will be noted that θ_i is measured from the positive direction of the x-axis, counter clock-wise, as in Trigonometry; and that by virtue of the definition of θ_i the signs of the components of the forces are provided for by the trigonometric functions. Consider for example P_3; the X-component and Y-component are both negative, as is the sign of $\cos\theta_3$ and $\sin\theta_3$. On the other hand, the X-component of P_4 is positive and the Y-component is negative. These correspond respectively to the sign of $\cos\theta_4$ and $\sin\theta_4$.

28. Theorem.—*For any system of forces acting on a point, there is one and only one resultant.*

As was shown in Art. 26, any system of concurrent forces may be replaced by two forces X, acting along the x-axis, and Y, acting along the y-axis.

These forces, and therefore the original system, could be replaced by two forces, X', acting along the line OX', and Y' acting along the line OY', which is perpendicular to OX', where, as may be seen by resolving along and perpendicular to these lines,

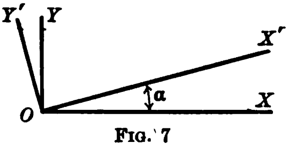


Fig. 7

$$\left.\begin{aligned} X' &= X\cos\alpha + Y\sin\alpha \\ Y' &= Y\cos\alpha - X\sin\alpha \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

where α is the angle between the lines OX and OX'.

Let R' be the resultant of X' and Y'. Then

$$\begin{aligned} R'^2 = X'^2 + Y'^2 &= X^2\cos^2\alpha + Y^2\sin^2\alpha + 2\,XY\sin\alpha\cos\alpha \\ &\quad + Y^2\cos^2\alpha + X^2\sin^2\alpha - 2\,XY\sin\alpha\cos\alpha \\ &= X^2 + Y^2 = R^2. \end{aligned}$$

That is, the magnitude of the resultant does not depend on the choice of the axes.

Let the direction of R' make an angle θ' with OX' and the direction of R make an angle θ with OX;
then, as in Art. 26,

$$\begin{aligned} \tan\theta' &= \frac{Y'}{X'} = \frac{Y\cos\alpha - X\sin\alpha}{X\cos\alpha + Y\sin\alpha} \\ &= \frac{R(\sin\theta\cos\alpha - \cos\theta\sin\alpha)}{R(\cos\alpha\cos\theta + \sin\alpha\sin\theta)} \\ &= \tan(\theta - \alpha). \\ \therefore\ \theta' &= \theta - \alpha. \end{aligned}$$

Therefore the direction of the resultant does not depend on the choice of axes, which proves the theorem.

29. The Determination of the Resultant of Forces Acting on a Particle, but not lying in the same Plane.

(1) *Graphical Determination.*

It is evident that by repeated applications of the parallelogram law, we may find the resultant of any number of forces whether or not they lie in one plane. The polygon of forces applies also in either case. However, if the forces are not all in a plane, the polygon will be *gauche*.

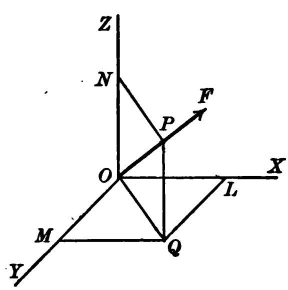


FIG. 8

(2) *Analytic determination.*

(*a*) *Any force may be resolved into three rectangular components.*

Let OX, OZ be at right angles, lying in the plane of the paper, and OY a line perpendicular to the plane XOZ. We shall call the plane YOZ the yz-plane and the XOY plane the xy-plane. We call the lines OX, OY, OZ, the x-, y-, and z-axes respectively; the positive direction of the x-axis being toward the right, of the z-axis

upward, and of the y-axis toward the reader. Let a force F act along the line OP. Let PQ be a perpendicular from P on the xy-plane, and QL be perpendicular to OX, and QM perpendicular to OY. Let OP represent the force F. It may be replaced by two forces represented by ON and OQ, where $ON = QP$. Then the force OQ may be replaced by forces represented by OL and OM. That is, the force F may be represented by three forces, OL, OM and ON. Q.E.D.

(*b*) Suppose now there be any number of forces, F_1, $F_2 \cdots$ acting at O, and not all in the same plane, making angles α_1, β_1, γ_1; α_1, β_2, γ_2; $\cdots$ with the x-, y-, and z-axes, respectively.

Then any force F_i can be replaced by three forces,

$$X_i = F_i \cos \alpha_i \text{ along the } x\text{-axis},$$
$$Y_i = F_i \cos \beta_i \text{ along the } y\text{-axis},$$
$$Z_i = F_i \cos \gamma_i \text{ along the } z\text{-axis}.$$

That is, the entire system of forces may be replaced by a series of forces acting along the x-, y- and z-axes.

We may add algebraically the forces acting along a given line.

$$\text{Let } X = \Sigma X_i = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + F_3 \cos \alpha_3 \cdots$$
$$\text{Let } Y = \Sigma Y_i = F_1 \cos \beta_1 + F_2 \cos \beta_2 + F_3 \cos \beta_3 \cdots$$
$$\text{Let } Z = \Sigma Z_i = F_1 \cos \gamma_1 + F_2 \cos \gamma_2 + F_3 \cos \gamma_3 \cdots$$

Then the entire system of forces has been replaced by three forces, acting along three lines at right angles to each other. Let R be the resultant of the three forces and therefore of the system; then

$$R^2 = X^2 + Y^2 + Z^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which determines the magnitude of R.

Moreover, if the angles that R makes with the x-, y-, and z-axes are respectively A, B, and C, then

$$R \cos A = X,$$
$$R \cos B = Y,$$
$$R \cos C = Z,$$

which determines the direction of R.

30. **Exercises**

1. A particle is acted upon by two forces, of 8 and 10 pounds respectively, making an angle of 30° with each other. Find the magnitude of the resultant.

2. Prove that the resultant of two equal forces bisects the angle between them.

3. Prove that the magnitude of the resultant of two forces acting in the same line equals their algebraic sum.

4. Prove that the magnitude of the resultant of any two forces not in the same straight line is less than the sum of the magnitudes of the forces, and greater than their difference.

5. The resultant of two forces of 8 and 10 pounds respectively, is 5 pounds. Find the angle between the forces.

6. Find the resolved part of a force of 10 pounds along a line making an angle of 45° with the force; making an angle of 90° with the force; making an angle of 120° with the force.

7. A particle is acted upon by three coplanar forces of 1, 2, and 3 pounds respectively. The 3-pound force makes an angle of 120° with the 1-pound force and an angle of 45° with the 2-pound force. Find, by drawing a force polygon, the resultant of the forces.

8. Find the sum of the resolved part of the forces along the direction of the 2-pound force. *Ans.* 3.16 (nearly).

9. A boat is being towed by two ropes making an angle of 60° with each other. The pull on one rope is 500 pounds, the pull on the other is 300 pounds. In what direction will the boat tend to move? What single force would produce the same result?

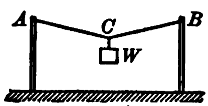
10. A particle is acted on by five coplanar forces; a force of 5 pounds acting horizontally to the right, and forces of 1, 2, 3, and 4 pounds making angles of 45°, 60°, 225°, and 300° with the 5-pound force. Find, using Art. 26, the magnitude and direction of the resultant. *Ans.* $R = 7.31$ (nearly), $\theta = 334° 28'$

approximately, where θ is the angle that the resultant makes with the force of 5 pounds.

11. In exercise 10, check the magnitude of the resultant, using the polygon of forces.

12. A man weighing 150 pounds stands on a ladder inclined 20° to the horizon. Find the resolved part of his weight along the ladder; perpendicular to the ladder.

13. A certain clothes line which is capable of withstanding a pull of 300 pounds, is attached to the ends A and B of two posts 40 feet apart; A and B being in the same horizontal line. When the rope is held taut by a weight W, attached to the middle point, C, of the line, C is four feet below the horizontal line AB. Find the weight of the heaviest boy it will support without breaking.



14. A street lamp weighing 100 pounds is supported by means of a pulley which runs smoothly on a cable supported at A and B, on opposite sides of the street. If A is 10 feet above B, and the street 60 feet wide, and the cable 75 feet long, find the point on the cable where the pulley rests, and the tension in the cable.

15. If the cable supporting the street lamp instead of being attached at A and B passed over smooth pulleys at those points and supported two weights of 100 pounds each, find the position in which the cable will come to rest.

16. A buoy is being towed by three boats. The first boat is moving due south and exerts a pull of 20 pounds; the second boat is moving S 15° E, and exerts a pull of 30 pounds; the third boat is moving S 20° W and exerts a pull of 18 pounds. What is the resultant force acting on the buoy and in what direction would the buoy move?

CHAPTER II

STATICS OF A PARTICLE

31. The study of mechanics is, for the sake of convenience, divided into two great subjects, viz.: *Statics* and *Dynamics*. Strictly, statics is the mechanics of bodies whose motion is not changing; that is, bodies moving with uniform velocity in a straight line, or, as a special case, bodies at rest. Our problem deals, almost exclusively, with bodies at rest.

32. Equilibrium. — A body whose motion is not changing is said to be in equilibrium. Also, the forces acting on such a body are said to be in equilibrium. The chief problem of Statics is to find the conditions of equilibrium. We shall begin by considering *the statics of a* **particle**, reserving for succeeding chapters the statics of rigid bodies.

33. It is necessary that the student understand, from the foregoing chapter, the following:

(*a*) A force is determined by its

1. Point of application.
2. Line of action and sense.
3. Magnitude.

(*b*) Law of parallelogram of forces, and its consequences.

34. Analytic Conditions of Equilibrium of Coplanar Forces Acting on a Particle. — From the Second Law of Motion it follows that one force will produce a change of motion. Hence, Statics is concerned with more than one force or with no forces. From the parallelogram law it follows that any number of forces acting on a particle can be replaced by a single force — their resultant. Hence a necessary and sufficient condition for

equilibrium of a system of forces acting on a particle is that their resultant be zero. Therefore, using the notation of Art. 26,

$$R = 0.$$

Therefore

$$X^2 + Y^2 = 0.$$

Since X^2 and Y^2 are positive quantities (X and Y being real), we conclude that

$$X = \Sigma X_i = 0, \text{ and } Y = \Sigma Y_i = 0. \quad . \quad . \quad . \quad (10)$$

Now the x- and y-axes may be any two lines at right angles to each other (Art. 28). We may therefore state the following theorem:

The necessary and sufficient conditions that any number of coplanar concurrent forces be in equilibrium are that the sum of the resolved parts of the forces along any two lines, at right angles to each other, and in the plane of the forces, is separately equal to zero.

This is a very important theorem.

It follows that:

If a system of forces be in equilibrium, the sum of the resolved parts of the forces along any line is zero.

It will be noted also that *either* $Y = 0$ or $X = 0$ is a necessary but not a sufficient condition for equilibrium.

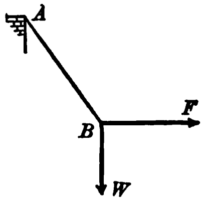
35. Analytic Conditions of Equilibrium of Non-Coplanar Forces Acting on a Particle. — Using the notation of Art. 29, it follows (Art. 34) that the necessary and sufficient conditions that any number of forces acting on a particle be in equilibrium are:

$$X = \Sigma X_i = 0,$$

$$Y = \Sigma Y_i = 0,$$

$$Z = \Sigma Z_i = 0.$$

36. **Exercises**



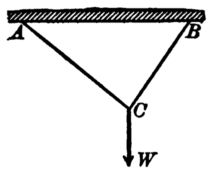
1. (*a*) A weight of 50 pounds suspended by a string AB is pulled aside by a horizontal force F so that AB makes an angle of 30° with the vertical. Find the tension T of the string and the horizontal force F.

(*b*) If AB makes an angle of θ with the vertical, prove that the greater θ is, the greater the tension in AB, W remaining constant. *Ans.* (*a*) $F = 28.9$ pounds; $T = 57.8$ pounds.

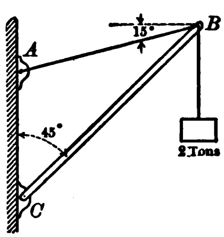
2. A and B are in a horizontal line. AC and BC are knotted to a ring that supports a weight of 50 pounds. Let

$$AB = 5 \text{ feet},$$
$$AC = 4 \text{ feet},$$
$$BC = 3 \text{ feet}.$$

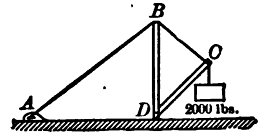
Find the tension in BC and AC.



3. Find a force that will equilibrate two forces of 6 pounds and 8 pounds, respectively, acting along lines making an angle of 45° with each other.

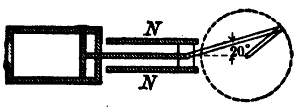


4. Find the tension in the cable AB and the compression in the jib BC, due to the load of 2 tons, supported at B. The angles which the members make with each other are shown in the sketch.

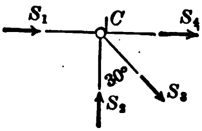


5. In the crane in the sketch, no member extends beyond the pin. BD is vertical and 12 feet long; AB is 15 feet long; BC, 7.2 feet; CD, 9.6 feet. Find the stresses induced by a load of 2000 pounds at C.

6. Steam in the cylinder of an engine exerts a pressure of 20,000 pounds on the piston head. The guides N are smooth. What is the thrust in the connecting rod when it makes an angle of 20° with the horizontal? What is the pressure on the guides N?



7. A child weighing 50 pounds is sitting in a swing, the seat being 12 feet below the point of support. What horizontal force would be required to hold the child 6 feet to the right of the vertical through the support? What would be the tension in the ropes?



8. Four members of a bridge truss act upon a pin at C. S_1, a horizontal stress of 125,000 pounds, is acting toward the right; S_2, a vertical stress of 25,000 pounds, is acting upward. Find the amount of the stresses S_3 and S_4, and determine whether or not they are correctly directed.

37. Conditions of Equilibrium of Concurrent Forces Considered Graphically. — It follows from Arts. 22 and 34 that a *necessary and sufficient condition of equilibrium is that the Force Polygon of the forces closes.* For in this case only is the resultant zero.

38. Triangle of Forces. — A special but very important case of the foregoing article is the one in which a particle is in equilibrium under the action of three forces. The polygon then becomes a triangle.

We have then the very useful theorem known as the Triangle of Forces, the statement of which follows:

If three forces acting on a particle are in equilibrium, the forces may be represented in magnitude and direction (but not line of action) by the sides of a triangle.

Three corollaries follow, proofs of which should be made by the student.

Cor. I. *If three forces acting on a particle are in equilibrium, they are in one plane.*

Cor. II. *If three forces acting on a particle are in equilibrium, any one of the forces is equal and opposite to the resultant of the other two.*

Cor. III. *The converse of the Triangle of Forces is also true, viz.: If three forces acting on a particle are represented in magnitude and direction (but not line of action) by the three sides of a triangle taken in order, they are in equilibrium.* This follows directly from Corollary II.

39. Let the particle at O be in equilibrium under the action of the three forces P, Q, and R. Construct the force triangle oab (Fig. 10).

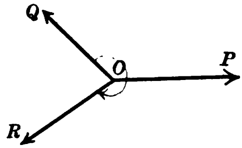


Fig. 9

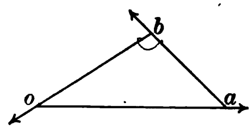


Fig. 10

Then $$\frac{P}{oa} = \frac{Q}{ab} = \frac{R}{bo} \text{ (by hypothesis).}$$

But $$\frac{oa}{\sin oba} = \frac{ab}{\sin boa} = \frac{bo}{\sin oab}.$$

$$\therefore \frac{P}{\sin QOR} = \frac{Q}{\sin POR} = \frac{R}{\sin POQ}. \quad . \quad . \quad (11)$$

That is, *the ratio of P to the sine of the angle between Q and R = the ratio of Q to the sine of the angle between P and R = the ratio of R to the sine of the angle between P and Q.*

40. Since many problems occur in practice in which three forces acting on a point are in equilibrium, these theorems are of great practical importance. In such cases equations (11) lend themselves to computation, while the theorems of Art. 38 are of special use in Graphic Statics. In order to apply equa-

tions (11) it is not necessary to construct two figures as is done in Art. 39. Consider, for example, the following problem:

A body whose weight, W, is suspended by a string which is fastened to two strings, AC and BC, at C, AB being horizontal.

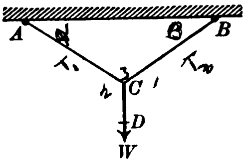


FIG. 11

Let angle $CAB = \alpha$ and angle $CBA = \beta$ be given; and let the tension in AC and BC be T_1 and T_2, respectively; to find the tensions T_1 and T_2.

Since the three forces T_1, T_2, and W, acting at the point C, are in equilibrium, we may at once write, using equation (11):

$$\frac{T_1}{\sin DCB} = \frac{T_2}{\sin DCA} = \frac{W}{\sin ACB};$$

or,

$$\frac{T_1}{\cos\beta} = \frac{T_2}{\cos\alpha} = \frac{W}{\sin(\alpha+\beta)}.$$

$$\therefore\ T_1 = \frac{W\cos\beta}{\sin(\alpha+\beta)};\ T_2 = \frac{W\cos\alpha}{\sin(\alpha+\beta)}.$$

41. **Exercises**

1. Solve problems 1, 2, 4, and 5, Art. 36, using the methods of Art. 40.

2. A particle is subject to three forces: A force of 10 pounds acting along the x-axis, a force of 50 pounds making an angle of 150° with the x-axis, and a force P. Find the direction and magnitude of P in order that the point may be in equilibrium.

Ans. 41.6 (nearly); $\theta = 323°\ 08'$ (nearly).

42. Types of Forces. — The student must keep clearly in mind, when we say that forces act on a body, that these forces represent the action of other bodies upon the body under consideration. Force is either a pull or a push, and can be measured by a properly placed spring balance, and any interaction of bodies that cannot be so measured is not a force. For example, a book lying on a table presses down on the

table with a force equal to the weight of the book, and also the table presses against the book with an equal force. A locomotive drawing a train exerts a pull on it which could be registered by a spring balance, and the train pulls with equal force on the locomotive in the opposite direction. These are mere illustrations of the third Law of Motion (Art. 11). It may be well to point out that the action and reaction do not act on the same body.

It will be helpful to enumerate here those forces that enter most frequently into problems of statics, and to state explicitly the assumptions usually made regarding the direction and other properties of these forces. The following are the most important:

The Weight (Art. 14) *of a Particle of a Body.* — This is the force with which the earth attracts a body toward its center, and acts therefore in a vertical line.

The Tension of a Cord or Cable. — The cords or cables which are met with in statics are all considered flexible, inextensible, and weightless. A flexible cord is one which may be bent without applying force.

The tension of a cord at any point P is the force with which a particle on one side of P acts on an adjacent particle

FIG. 12

on the other side of P. The tension of a cord at any point is along the direction of the cord at that point.

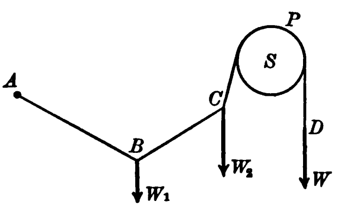


FIG. 13

For example, suppose one extremity of a cable $ABCPD$ (Fig. 13) is fastened at A, that the cord supporting W_1 is attached to a smooth ring or pulley, and the string supporting W_2 is *knotted* to the cable at C; that the cable passes over

a smooth cylinder or pulley S, and supports a weight W. Then the tension of the string at any point along the cable AB is in the direction AB; while the tension of the cable at any point P, touching the cylinder S, is along a tangent to the cylinder at that point. The magnitude of the tension is constant between any two points to which external forces are applied, provided that the points where the forces are applied to the cable cannot slip along the cable. Under the above assumptions the tension in the string CPD is constant throughout its length and is equal to W. Moreover, since the string which supports W_1 is attached to a *smooth* ring through which the string ABC passes, the tension in ABC is constant throughout its length.

Reaction of Surfaces. — Another way in which force can be applied to a particle is by the pressure between a particle and a rigid body with which it is in contact. Such a force is called a reaction. The pressure of the particle on the body is equal and opposite to that of the body on the particle. For the present we shall content ourselves with the reaction of *smooth* curves and *smooth* surfaces. By a smooth curve or a smooth surface we shall mean one which gives a reaction normal to the curve or surface. This gives the *direction* of the force. For example, a particle lying on an inclined plane is acted on by three forces; W, its weight, which is vertical; R, the reaction of the plane, which is perpendicular to AB; and some other force, P.

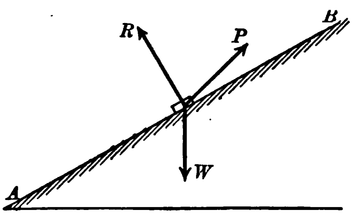


FIG. 14

Reaction of Pins. — Many structures, such as bridge or roof trusses, are made up of bars (such as AB, BC, etc., in Fig. 15) which are connected at their extremities by means of pins. We shall assume that these bars, or members as they are usually called, are rigid bodies, that is, do not change their

shape, and for the present, that they are without weight. By means of some mechanical arrangement such as is shown in Fig. 15, the load which these trusses carry is concentrated

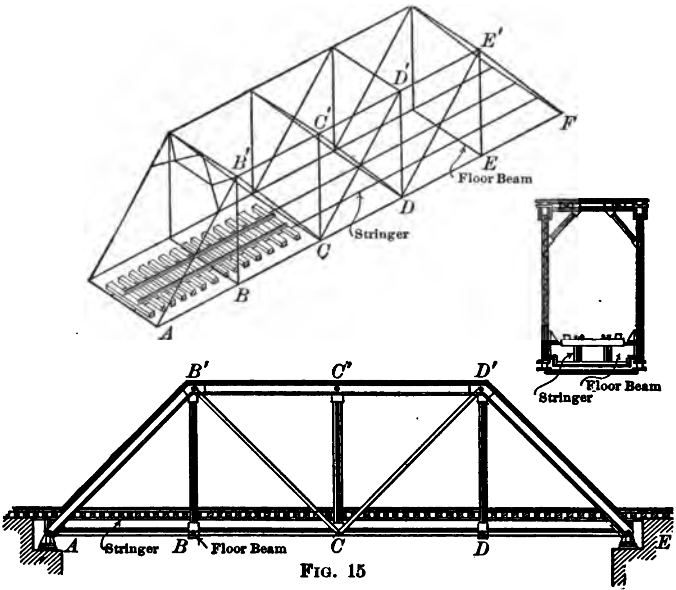


FIG. 15

at the joints. Therefore, the only actions to which the members are subjected are the pressures brought by the pins at each end of the bar. In general, when a pin passes through a bar, the direction of the pressure of the bar upon the pin is unknown. In the case of the structures under consideration, each bar is acted upon by two forces only; hence, for equilibrium, these forces must be equal and opposite and have the same line of action. The line of action of the pressure of the bar on the pin is fixed, therefore, by the points of contact of the pin and the bar at either extremity, and is always taken as the line passing through the centers of the pins. Therefore, the direction of each of the pressures on a pin is given by the axes of the members through which the pin passes.

For example, let us consider the member AB of Fig. 15. The line of action of the stress in AB is the line through the centers of the pins at A and B. If the pressure of the pin at A is toward B, the direction of the pressure of the pin at B must be toward A, and the bar is said to be in compression. If the pressure of the pin at A is away from B, the pressure of the pin at B must be away from A, and the member is said to be in tension. Furthermore, these pressures must be equal or the bar would move.

Friction, Attraction, and Repulsion. — These will be considered in later chapters.

43. Applying the concluding remarks concerning bars in the preceding article, the mechanics of the particle is easily extended to the mechanics of a finite number of points rigidly connected. For example, having found the reaction of the pin at A on the bar AB (Fig. 15), we know at once the reaction of the pin at B against this particular bar, and by considering the equilibrium of the pins, one at a time, we may in general get the tension or compression in every member of the truss. This method is illustrated by problems 4 and 5, Art. 36. This is done in the graphical solutions of roof and bridge trusses. Simpler analytical methods for bridge trusses will be presented later.

44. Remark. — We have, in the last articles, defined a number of fictitious bodies, such as flexible, weightless strings, smooth surfaces, weightless bars, etc. This is theoretically wise because the mechanics of these fictitious bodies is more simple than that of those occurring in nature, and because, having developed the theorems for these bodies, it is not difficult to extend them so as to apply to practical conditions. This we shall do in later chapters. Moreover, further study will convince the reader that these conditions are, in the main, not as far from the truth as one might at first suppose. Compared with the loads that a bar or cable carries in practical problems, the weight is often negligible. An additional justification of this practice is the fact that the results thus obtained conform with experience. With few exceptions we have limited our discussion to coplanar forces. A little reflection will convince one that nearly all problems concerning statical structures, such as bridges, etc., are, or can be reduced to, problems of this type.

45. Solution of Problems. Suggestions. — Skill acquired by practice counts for a great deal. Make a drawing, indicating by a line and arrowhead the line of action and sense of every force acting on the particle whose equilibrium is being considered. For any particle *there are two, and only two, independent equations of equilibrium* which will determine two unknowns. If there are n points rigidly connected, we may write $2n$ independent equations.

It may be that certain geometrical conditions are given. But if there is a greater number of conditions than unknown quantities, certain relations must exist between the quantities or the problem is impossible. If there is a fewer number of conditions than unknown quantities, a solution is impossible. No specific directions can be given. In a general way, when three forces act at a point, apply the Triangle of Forces. When there is a greater number, resolve forces along lines chosen arbitrarily. One can usually shorten his solution by resolving the forces along a line at right angles to a force whose value he does not wish to find.

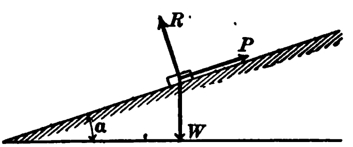


FIG. 16

For example:

A particle of weight W, Fig. 16, lies on a smooth plane which makes an angle α with the horizon. Find the force P acting along the plane required to keep the particle in equilibrium, and the reaction of the plane.

The particle is in equilibrium under the action of three forces, P, W, and R, the reaction of the plane.

Resolve the forces along the plane and perpendicular to the plane. Then:

$$W \sin \alpha = P,$$
$$W \cos \alpha = R.$$

These are the only independent equations. From them we can find any two of the quantities if the remainder are given.

Or, we may apply the Triangle of Forces. Then:

$$\frac{W}{\sin 90°} = \frac{R}{\sin (90° + \alpha)} = \frac{P}{\sin (180° - \alpha)}.$$

$$W = \frac{R}{\cos \alpha} = \frac{P}{\sin \alpha}.$$

A Second Example. — Consider the arm of a cantilever conveyor acted upon by forces as indicated in Fig. 17. Let it be required to find the tension or compression in each member of the cantilever arm. The members are considered weightless, are connected by pins, and none of them extends beyond a joint.

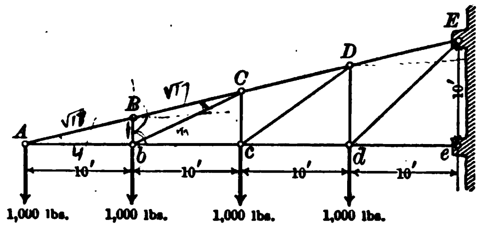


FIG. 17

Let us first consider the pin at A.

There are three forces acting upon this pin: the load equal to 1000 lbs., the unknown stress in Ab, which we will call T_1, and the unknown stress in AB, which we will call T_2. As we do not know the sense of T_1 and T_2, we will direct them away from A.

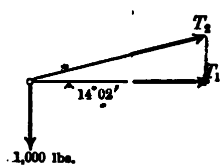


FIG. 18 *a*

Resolving horizontally,

$$T_2 \cos 14° \, 02' + T_1 = 0.$$

Resolving vertically,

$$T_2 \sin 14° \, 02' - 1000 = 0.$$

$$\therefore \; T_2 = \frac{1000}{\sin 14° \, 02'} = \frac{1000}{.2425} = 4120 \text{ lbs. (nearly).}$$

$$T_1 = -\,4120 \cos 14° \, 02' = -4120 \times .9701$$
$$= -\,4000 \text{ lbs. (nearly).}$$

Inasmuch as we know that we have taken the sense of the 1000 lbs. correctly, the negative sign of T_1 shows that we have

chosen the sense of T_1 opposite to that in which it really acts; while the positive sign of T_2 shows that it acts in the direction chosen.

Since T_1 acts toward the pin, the member Ab is in compression. Similarly, since T_2 acts away from the pin, the member AB is in tension.

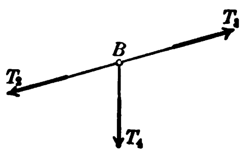


FIG. 18 b

We may now consider the pin at B, since only two of the forces acting on it remain unknown. Again there are three forces acting: the known tension in AB, T_2; the unknown stress in BC, T_3; and the unknown stress in Bb, T_4. Again we will indicate that the unknown forces are acting away from the pin.

Resolving normally to the upper chord,

$$T_4 \cos 14° \, 02' + 0 = 0.$$
$$\therefore T_4 = 0.$$

Resolving along the upper chord,

$$+ T_3 - T_2 = 0.$$
$$\therefore T_3 = T_2 = 4120 \text{ lbs. (nearly)}.$$

Evidently under the loading indicated the member Bb would get no stress. It would be stressed, however, if any load were applied at B.

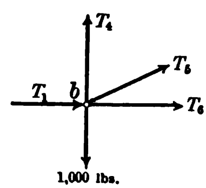


FIG. 18 c

Let us now consider the pin at b. There are five forces acting upon this pin, three of which are known. The known forces are: the load, equal to 1000 lbs.; T_1, equal to 4000 lbs.; and T_4, equal to zero. The unknown forces are the stresses in bC and bc, which will be designated by T_5, and T_6, respectively, and will be directed away from the pin.

Resolving horizontally,

$$T_6 + T_5 \cos 26° \, 34' + T_1 = 0.$$

Resolving vertically,

$$T_5 \sin 26° \, 34' + T_4 - 1000 \text{ lbs.} = 0.$$

$$T_5 = \frac{1000}{\sin 26° \, 34'} = \frac{1000}{.447} = 2240 \text{ lbs. (nearly)}.$$

$$T_6 = -\,2240 \cos 26° \, 34' - 4000 = -\,6000 \text{ lbs. (nearly)}.$$

Again the negative sign indicated that the pressure brought by the bar bc is toward the pin. This general statement can now be made. If the unknown stresses are always directed away from the pin, a positive sign for an unknown stress indicates that that member is in tension, and a negative sign that it is in compression.

The student may find the stress in the members CD and Cc by considering the equilibrium of the pin at C. He will note that the pin at C must be considered next, for if the pin at c were taken, there would be three unknown forces with only two conditions of equilibrium. Thus, by considering each pin in turn, the stress in each member of the cantilever arm can be obtained.

46. Exercises

1. Two forces of 10 lbs. and 20 lbs. act on a particle. Find the resultant:

(*a*) When the angle between them is 60°.

(*b*) When the angle between them is 150°.

2. The resultant of two forces of 6 lbs. and 5 lbs. is 10 lbs. Find the angle between them.

3. Five coplanar forces of 1 lb., 2 lbs., 3 lbs., 4 lbs., and 5 lbs. act on a particle. The first, second, third, and fourth make in order angles of 30°, 45°, 150°, and 240° with the fifth. Find the magnitude of the resultant and the angle it makes with the fifth force.

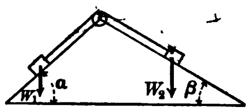
4. The resultant of two forces P and Q acting on a particle bisects the angle between them. Prove that $P = Q$. Find R.

5. A body of 50 lbs. is at rest on a smooth plane inclined at an angle of 30° to the horizontal. It is acted on by a horizontal force P. Find P.

6. A particle whose weight is W rests on a smooth inclined plane. It is acted on by a horizontal force $\frac{1}{3}W$ and by a force $\frac{W}{3}$ upward, parallel to the plane. Find the inclination of the plane.

Ans. — If $\alpha =$ inclination of plane, $\alpha = 2\tan^{-1}(\frac{1}{3})$.

7. W_1 and W_2 rest on a double inclined smooth plane and are connected by a string which passes over a smooth pulley so that the parts on the string are parallel to the planes. Prove $W_1 \sin\alpha = W_2 \sin\beta$.



8. A particle weighing W tons lies on a smooth plane inclined 45° to the horizon; the particle is prevented from slipping by a string which is inclined 30° to the plane and (a) above the plane; (b) below the plane; the string passes over a smooth pulley and supports a body weighing P tons. Find P, the reaction of the plane, and the tension of the string.

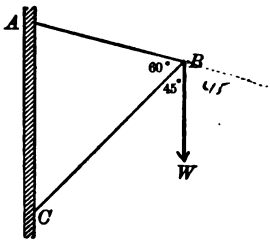
Ans. (a) $P = \frac{\sqrt{6}}{3}W$. The reaction of the plane $= W\frac{(\sqrt{3}-1)}{\sqrt{6}}$.

9. A weight W is suspended from two fixed points A and B by a string ACB, C being the point of the string where W is attached. If AC and BC are inclined to the vertical by the given angles α and β, respectively, find the tension of AC and BC.

10. A weight W is supported by a string attached to two strings AC and BC, AB being inclined 10° to the horizon, A lower than B. If $AB = 6$ feet, $AC = 3$ feet, and $BC = 5$ feet, find the tension in AC and BC.

Ans. $T_{AC} = .7686\,W$; $T_{BC} = .693\,W$.

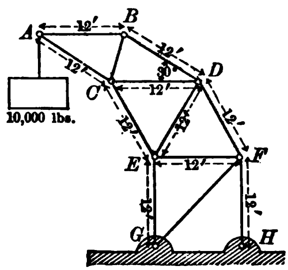
11. Resolve a force of 20 lbs. into two components making angles of 45° and 30° with the given force.



12. AB is a cord attached to BC, a rigid bar, fastened by a pin at C to a wall AC, as in the figure. W is suspended by a cord attached to B. Find the tension in AB and pressure in BC.

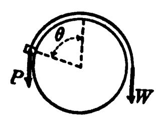
The elements of the crane are involved in this problem.

Ans. — $T_{AB} = \frac{\sqrt{2}}{\sqrt{3}} W$; $C_{BC} = \frac{\sqrt{3}+1}{2\sqrt{2}} W.$



13. Find the stresses in members CD, Dd, dE, and de of the cantilever arm shown in Fig. 17.

14. Find the stresses in the cantilever crane in figure due to a load of 10,000 lbs. applied at A.

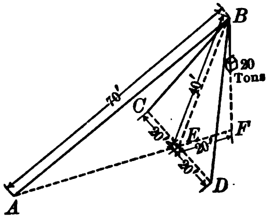


15. A body weighing W pounds is supported by a string which, passing over a smooth circle, is attached to a body weighing P pounds, lying on the convex side of the circle. Find the angle that the radius joining P to the center of the circle makes with the vertical.

16. Two particles whose weights are W_1 and W_2, respectively, lie on the concave side of a smooth vertical circle. They are joined by a stiff weightless rod whose length subtends at the center of the circle an angle $= 2\,\alpha$. Find the angle θ that the rod makes with the vertical diameter of the circle. Prove that the lighter particle lies higher than the heavier one. Find the thrust on the rod and the normal reaction of the circle.

17. The two smooth sides of a trough making an angle of 90° with each other support a smooth cylinder weighing 500 pounds. If one side of the trough makes an angle of 30° with the horizontal, what are the pressures between the cylinder and the trough?

18. A pair of sheer-legs is being used to raise a load of 20 tons from the hold of a vessel to its deck. What is the tension in the member AB, and the thrust in the members BC and BD, when the sheers are in the position shown? BC and BD are of equal length. See figure for other dimensions. Neglect the weights of the members.



Hint. — First consider the members BC and BD replaced by a single member BE, and find the stresses in the members AB and BE. Then find the thrusts in BC and BD as if BCD were vertical and acted upon by a vertical force equal to the stress in the fictitious member BE.

19. What are the stresses in each of the above members when the pair of sheers is lifting 50 tons, the distance EF being 5 feet?

CHAPTER III

FORCES ACTING ON A RIGID BODY

47. We shall consider the action of forces on rigid bodies only. A *rigid body* is one such that the distance between any two of its points is constant. In nature there are no rigid bodies, most bodies being deformed more or less under the action of a system of forces. However, it is true that given a system of forces, most bodies assume a certain form and retain it so long as that system acts; that is, for that system of forces the bodies are rigid. Take, for example, a weight W supported by two strings AB and CB. So long as W is unchanged and A, B, and C are fixed, the body ABC satisfies our definition of a rigid body.

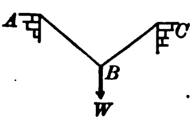


FIG. 19

48. We shall now make two additional assumptions:

(*a*) The effect of a force on a rigid body is unchanged if its point of application is transferred to any point rigidly connected to the body and in the line of action of the force. The principle assumed in this statement is known as the *transmissibility of forces.*

(*b*) The effect on a rigid body is unchanged if two equal and opposite forces, acting at any point of the body, are removed or introduced.

49. Non-parallel Forces. — If a rigid body M be acted upon by two non-parallel forces P and Q acting respectively at A and B, we may, (Art. 48), consider the forces as acting at D, the point of intersection of the lines of action of the forces. Hence, we may find their resultant by the Parallelogram Law.

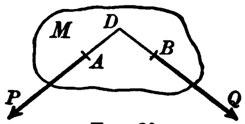


FIG. 20

If D falls without the body M, we may think of it as being connected rigidly with M by means of weightless, rigid rods.

If any number of forces P, Q, S, . . . act on the body M, we can find the resultant of two of them, then the resultant of this resultant and one of the remaining forces, and so on. Hence, by the method of this article we may find the resultant of any number of non-parallel forces acting on a rigid body if in the process it does not become necessary to find the resultant of two parallel forces. This method is generally impracticable and presently we shall develop one more easily applied.

50. Resultant of Parallel Forces. — Let us now find the resultant (Art. 19) of two parallel forces acting on a rigid body. There are two cases.

Case I. The parallel forces act in the same direction (Fig. 21 a).

Case II. The parallel forces act in opposite directions (Fig. 21 b). The demonstration which follows applies to both cases, until we reach equation (12).

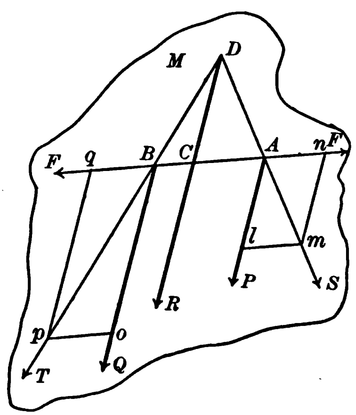


FIG. 21 a

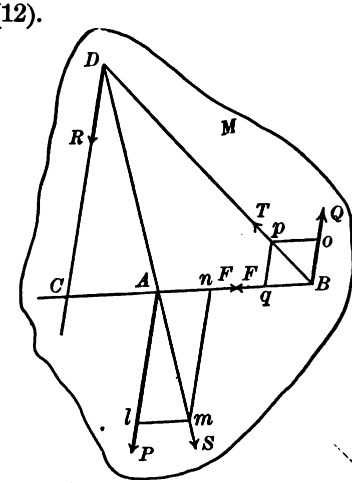


FIG. 21 b

Let two parallel forces P and Q act on the rigid body M, their points of application being respectively A and B. Join AB. Let P be represented by Al and Q by Bo. At A and B introduce two equal and opposite forces F; the effect on the body is unchanged (Art. 48). Let F be represented by An and $-F$ by Bq. Complete the parallelograms $Almn$ and $Bopq$. The effect on the body is unchanged if instead of the forces F and P at A we substitute their resultants, represented by Am, and instead of $-F$ and Q we substitute their resultant T, represented by Bp, and transfer the point of application of each of these forces to D, the point of intersection of their lines of action. We may now resolve S into two components, one of which is equal and parallel to F; the other component is equal in magnitude and parallel to P. Similarly, we may replace T by $-F$ and Q. We shall then have acting at D four forces, two equal and opposite (the forces F) which annul each other, and two forces P and Q acting in the same line which is parallel to the line of action of the original forces. These forces may be replaced by a single force R, acting along the line DC parallel to the original forces and equal in magnitude to the algebraic sum of P and Q. We may regard R as acting at C.

Now, $$\frac{DC}{AC}=\frac{Al}{An} \text{ (similar triangles)}$$
$$=\frac{P}{F} \text{ (by construction).}$$

Similarly, $$\frac{DC}{BC}=\frac{Bo}{Bq}=\frac{Q}{F}.$$

Therefore, $$\frac{AC}{BC}=\frac{Q}{P},$$

or $$P\cdot\overline{AC}=Q\cdot\overline{BC}. \quad \dots\dots\dots \quad (12)$$

The equation (12) is an important one.

For Case I, $$BC=AB-AC.$$

Therefore, $$AC=\frac{Q\cdot\overline{AB}}{P+Q}, \quad \dots\dots\dots \quad (13)$$

which determines C.

For Case II, $BC = AB + AC.$

Therefore, $$AC = \frac{Q \cdot \overline{AB}}{P - Q}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (14)$$

which determines C except when $P = Q$.

It will be observed that equation (13) and equation (14) give unique results. However, when $P = Q$, equation (14) fails — a case which we shall discuss later. (See Art. 60.) In Case I, C lies between A and B. In Case II, it lies without the segment AB. In either case, C is nearer the line of action of the greater force.

51. We have, therefore, the very important

THEOREM. — *The resultant of two parallel forces P and Q (P and Q not equal and opposite) acting respectively at A and B is parallel to the given forces; its magnitude equals the algebraic sum of the forces; its line of action cuts AB in a point C such that*

$$R \cdot \overline{AC} = Q \cdot \overline{AB}.$$

In order to find the resultant of any number of parallel forces it is only necessary to apply repeatedly the process of the last two articles.

52. The point of application of the resultant of any system of parallel forces is called the *center of the parallel forces.* It is customary to define its position by means of its rectangular coördinates.

53. To find the center of any number of parallel forces.

THEOREM. — *Let P_1, P_2, P_3, $\cdots$ be a system of parallel forces acting on a rigid body at the points whose coördinates are respectively x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3; $\cdots$ referred to OX, OY, and OZ as axes.*

Let $\bar{x}$, $\bar{y}$, $\bar{z}$, be the coördinates of the center of the system, then (except when $\Sigma P = 0$)

$$\bar{x} = \frac{P_1x_1 + P_2x_2 + P_3x_3 \cdots}{P_1 + P_2 + P_3} = \frac{\Sigma Px}{\Sigma P} \quad \cdot \quad \cdot \quad \cdot \quad (15)$$

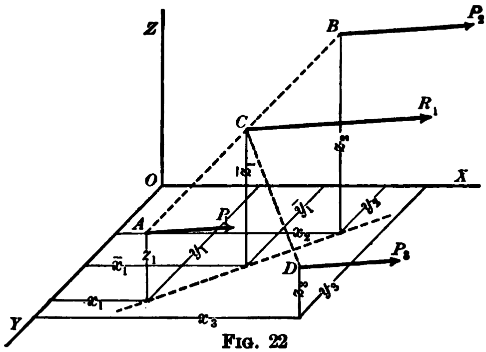


FIG. 22

and $$\bar{y} = \frac{P_1y_1 + P_2y_2 + P_3y_3 \cdots}{P_1 + P_2 + P_3} = \frac{\Sigma Py}{\Sigma P} \quad . \quad . \quad . \quad . \quad (16)$$

and $$\bar{z} = \frac{P_1z_1 + P_2z_2 + P_3z_3 \cdots}{P_1 + P_2 + P_3} = \frac{\Sigma Pz}{\Sigma P}. \quad . \quad . \quad . \quad . \quad (17)$$

Proof.—Let coördinates of

A, the point of application of P_1, be x_1, y_1, z_1,
B, the point of application of P_2, be x_2, y_2, z_2,
C, the point of application of R_1, be $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$,

where R_1 is the resultant of P_1 and P_2.

Then, $$\frac{AC}{BC} = \frac{P_2}{P_1} \text{ (from equation (12)).}$$

But $$\frac{AC}{BC} = \frac{\bar{x}_1 - x_1}{x_2 - \bar{x}_1} \text{ (similar triangles).}$$

$$\therefore \frac{\bar{x}_1 - x_1}{x_2 - \bar{x}_1} = \frac{P_2}{P_1}.$$

Hence, $$\bar{x}_1 = \frac{P_1x_1 + P_2x_2}{P_1 + P_2}.$$

In a similar way we may show that

$$\bar{y}_1 = \frac{P_1y_1 + P_2y_2}{P_1 + P_2}; \quad \bar{z}_1 = \frac{P_1z_1 + P_2z_2}{P_1 + P_2}$$

The student will find this a useful exercise.

From this theorem we may say that $\bar{x}_2, \bar{y}_2, \bar{z}_2$, the coördinates of the point of application of R_2, the resultant of R_1 and P_3, are

$$\bar{x}_2 = \frac{R_1\bar{x}_1 + P_3x_3}{P_1 + R_3} = \frac{(P_1 + P_2)\left(\frac{P_1x_1 + P_2x_2}{P_1 + P_2}\right) + P_3x_3}{(P_1 + P_2) + P_3}$$

$$= \frac{P_1x_1 + P_2x_2 + P_3x_3}{P_1 + P_2 + P_3}.$$

Similarly, $\bar{y}_2 = \dfrac{P_1y_1 + P_2y_2 + P_3y_3}{P_1 + P_2 + P_3}$ and $\bar{z}_2 = \dfrac{P_1z_1 + P_2z_2 + P_3z_3}{P_1 + P_2 + P_3}$.

54. Remark. — By repeated application of this process we could find the coördinates of the point of application of the resultant of any number of parallel forces, and we should find that the equations (15), (16), and (17) are always true.

To prove this statement: Assume the theorem is true for $n - 1$ forces. Let $\bar{x}_{n-1}, \bar{y}_{n-1}$ be coördinates of their resultant.

Then

$$\bar{x}_{n-1} = \frac{P_1x_1 + P_2x_2 + \cdots P_{n-1}x_{n-1}}{P_1 + P_2 + P_3 + \cdots P_{n-1}} \text{ (by hypothesis).}$$

Then, the resultant of these forces and another force P_n applied at the point x_n, y_n, will have for the coördinates of its point of application

$$\bar{x} = \frac{(P_1 + P_2 + \cdots P_{n-1})\frac{(P_1x_1 + P_2x_2 + \cdots P_{n-1}x_{n-1})}{(P_1 + P_2 + \cdots P_{n-1})} + P_nx_n}{P_1 + P_2 + P_3 + \cdots P_n}.$$

$$= \frac{\Sigma Px}{\Sigma P},$$

which proves the proposition for any number of forces. For since it holds true for two forces, (Art. 53), by this article it holds for three forces; therefore for four, and so on.

55. It will be noted (from equations (15), (16), and (17)) that, except in the case $\Sigma P = 0$, there is one, and only one, center of any number of parallel forces. But that in case $\Sigma P = 0$, there is no resultant with its point of application at a finite distance from the origin.

It will be noted that the coördinates of the center of parallel forces are independent of the direction of the forces.

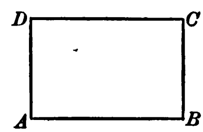
56. **Exercises**

1. Four particles weighing respectively 1, 2, 3, and 4 pounds, lie on a bar 3 feet long and without weight. The one-pound weight and four-pound weight lie at the opposite ends of the bar, the two-pound weight is one foot from the one-pound weight, and the three-pound weight, one foot from the four-pound weight. Where can a single force be applied to support the system? What is the magnitude of the force?

2. A one-pound weight lies on each of three corners of a square without weight. From what point in the square can it (the square) be supported in a horizontal position by a single string?

3. Two men carry 20 pounds attached to a bar 5 feet long without weight. If the body is two feet from one man and three feet from the other, how much does each carry?

4. Four parallel forces act on a plane rectangle. The forces and the coördinates of their point of application referred to AB and AD as axes are respectively, 150 pounds at (1, 1), 100 pounds at (5, 3), 200 pounds at (0, 4), and 300 pounds at (4, 0). Find their resultant and the point of its application.



5. Three men are carrying a weight of 150 pounds suspended by a string from a weightless rod 12 feet long. If two men are at one end of the rod and one at the other, where must the weight be placed in order that each man shall carry the same weight?

6. In the horizontal triangle ABC, $AB = 3$ feet, $BC = 4$ feet, and $CA = 5$ feet. If the triangle is weightless, and there are weights of 3 pounds at A, 4 pounds at B, and 5 pounds at C, find the point at which a single string will support the triangle in a horizontal position.

7. A truck with its load weighs 4000 pounds. The axles are 8 feet apart and each carries its share of the load. At a

certain position, when the load is crossing a bridge 20 feet long composed of two parallel beams, the pressure under the ends of the beams toward which the truck is moving, due to the weight of the truck, is 1600 pounds. How far are the front wheels from the forward end of the bridge?

57. The Moment of a Force. Definitions. — *The moment of a force with respect to a point* is the product of the magnitude of the force and the perpendicular distance of the point from the line of action of the force. Let O be any point, F any force whose magnitude is F, p be the perpendicular distance of O from the line of action of F, then the moment of F with respect to O equals $F \cdot p$. The perpendicular p is called the *arm* of the force.

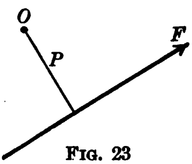


FIG. 23

The point O is called the *center of moments.* The plane determined by O and the line of action of the force is called the *plane of moments.* A line perpendicular to the plane of moments is called the *axis of moments.* The moment of a force with respect to an axis of moments is the product of the component of the force perpendicular to the axis and the perpendicular distance between the line and the line of action of the force. Hence, the moment of a force with respect to a point is the moment of force with respect to the axis of moments through that point.

58. Sign of a Moment. — We shall for the present assume that the effect of a moment is to tend to produce rotation around the axis with respect to which the moment is taken. If the tendency of the moment is to turn the body counter-clockwise, it is said to be positive; if clockwise, it is said to be negative.

59. The Composition of Moments.

THEOREM. — *The algebraic sum of the moments of two coplanar forces with respect to any point in their plane equals the moment of their resultant with respect to that point.*

There are two cases.

Case I. The lines of action of the forces are not parallel.

Proof. — Let the body M be acted upon by two forces P and Q represented by the sides of a parallelogram, and let R be their resultant. To find the moments of these forces about any point O in the plane of the forces.

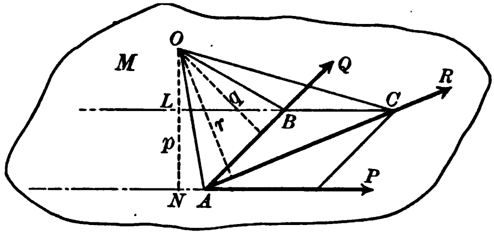


Fig. 24

Let the perpendicular distance from O on P, Q, and R be respectively p, q, r. From the figure,

$$\triangle\, OCA = \triangle\, OAB + \triangle\, BAC + \triangle\, OBC.$$

$$\therefore Rr = Qq + P(\overline{OL} + \overline{LN})$$

$$= Qq + Pp.$$

That is, Rr, the moment of R with respect to O, equals Pp, the moment of P with respect to O, plus Qq, the moment of Q with respect to O.

The student may prove that if O lies within the angle PAQ, or its vertical angle, that

$$Rr = |Pp - Qq|.$$

Case II. The lines of action of the forces are parallel.

Note. — We shall exclude the case where the forces are equal and opposite.

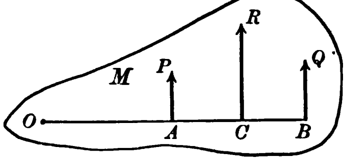


Fig. 25

Let P and Q, two forces, act on the body at A and B, and R, their resultant, act at the point C. From O, any point, draw a perpendicular $OACB$ to the lines of action of the forces.

The moment of P with respect to $O = P \cdot \overline{OA}$.
The moment of Q with respect to $O = Q \cdot \overline{OB}$.
Now, $P \cdot \overline{OA} + Q \cdot \overline{OB} = P(\overline{OC} - \overline{AC}) + Q(\overline{OC} + \overline{CB})$

$$= (P + Q)\overline{OC} \text{ (since } P \cdot \overline{AC} = Q \cdot \overline{BC})$$
$$= R \cdot \overline{OC}. \qquad \text{Q.E.D.}$$

The student may prove the theorem in case P and Q act in opposite directions; also in the case O lies between A and B.

60. Couple. — In Art. 50, we showed that if two forces P and Q acting on a rigid body are parallel and equal and opposite in direction, equation (14) fails, and that no resultant of P and Q acting at a finite distance, exists. We shall now consider this case.

Two *equal, parallel,* and *opposite forces* acting on a rigid body constitute a *couple.* If the forces act on the same point, they will produce equilibrium; therefore we shall consider only those forces whose lines of action are not coincident. The perpendicular distance between the two forces is called the *arm* of the couple. The plane determined by the lines of action of the two forces is called the *plane* of the couple.

61. THEOREM. — *The moment of the forces of a couple with respect to any point in the plane of the couple is constant and equal to the product of one of the forces and the arm of the couple.*

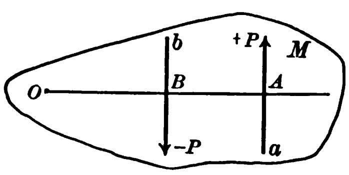


FIG. 26

The forces of the couple P and $-P$ act on the body M at the points a and b, respectively. Through O, any point, draw a line OBA perpendicular to the lines of action of the forces. If C be the moment of the forces with respect to O, then

$$C = P \cdot \overline{OA} - P \cdot \overline{OB} = P \cdot \overline{AB}. \qquad \text{Q.E.D.}$$

62. THEOREM. — *The effect of a couple on a rigid body is unchanged if it be replaced by another couple in the same or a*

parallel plane having the same moment. There are three steps in the proof, which follows.

(*a*) We may replace the given couple by another couple having its arm and forces equal and parallel to those of the original couple.

Let P and $-P$ act on a rigid body at A and B, respectively. Draw $A'B'$ parallel to and equal to AB, and at each of the points A' and B' introduce two forces $+P$ and $-P$, each parallel to P and of the same magnitude. Then $+P$ at A and $+P$ at B' have for their resultant $+2P$, acting at C; and $-P$ at A' and $-P$ at B have for their resultant $-2P$, acting at C. Since these forces annul each other, there remain only two forces, $+P$ at A' and $-P$ at B'; that is, *without changing the mechanical conditions of the body* we have replaced a couple by another equal couple acting in a parallel plane.

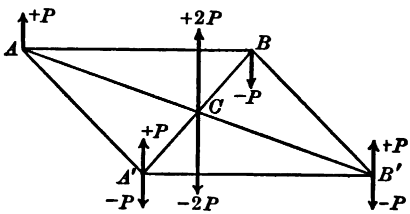


FIG. 27

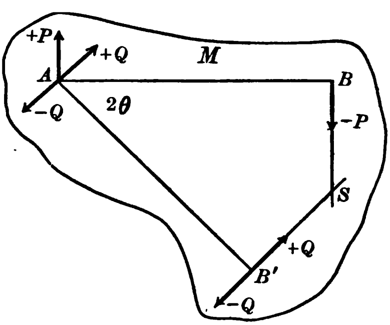


FIG. 28

(*b*) The couple (*i.e.* arm and line of action of the forces) may be rotated through any angle in its plane.

Through A draw AB' equal to AB and making an angle of 2θ with it. At A and at B' introduce two forces Q and $-Q$, equal in magnitude to P. The resultant of $+P$ and $-Q$ at A equals the resultant of $-P$ and $+Q$ at S, and acts in the same line and in the opposite direction.

There remains, therefore, a couple Q (which is equal to P) at A and $-Q$ at B'.

(c) We may replace the couple by another couple having the same moment.

Let P acting at A and $-P$ acting at B be the forces of the given couple. At A' and at B' introduce forces Q and $-Q$ parallel to P, choosing Q so that

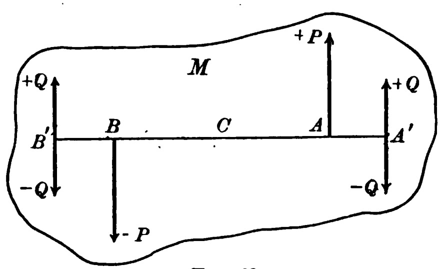


FIG. 29

$$Q \cdot \overline{A'B'} = P \cdot \overline{AB}.$$

By (a) move $A'B'$ along AB until their middle points coincide at C. The resultant of $+Q$ at B' and $+P$ at A equals $+(P+Q)$ and acts at C, for

$$Q \cdot \overline{B'C} = P \cdot \overline{AC}.$$

Similarly, $-Q$ at A' and $-P$ at B have as a resultant $-(P+Q)$ acting at C. Hence, there remains the couple $+Q$ acting at A' and $-Q$ acting at B' having the same moment as the original couple.

63. Composition of Couples.

THEOREM. — *Any number of couples is equivalent to a single couple.*

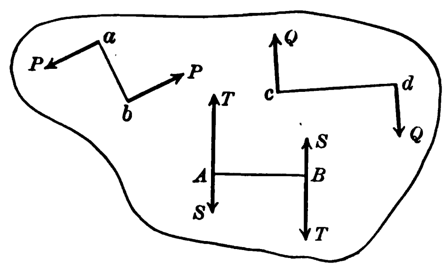


FIG 30

(a) The couples are in the same or parallel planes. Consider two couples: the one with forces P and arm ab, the other with forces Q and arm cd. Substitute for the P-couple, another with forces S and arm AB where

$$S \cdot \overline{AB} = P \cdot \overline{ab}.$$

Similarly, replace the Q-couple by another with forces T and arm $\overline{AB}$. At A there is acting a system of forces in the same line, the resultant of which equals the algebraic sum of S and T. The resultant of the forces at B is equal, parallel, and opposite to those at A. That is, the result of the substitution is a couple. The process applies equally well for any number of couples.

(*b*) The couples are in intersecting planes. By Art. 62 we can transfer the couples so that the arms of the couples coincide with the intersection of their planes and by (*a*) of the present article, we can make the arms of the same length.

Let the arm of the resulting couples be AB and the forces of the couples be P and Q, respectively. Then the resultant of P and Q at A is R by the Parallelogram Law; the resultant of $-P$ and $-Q$ at B will be a force of the same magnitude, let us call it $-R$.

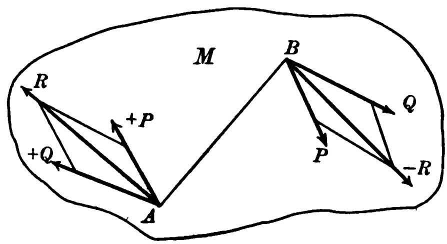


FIG. 31

Now, the plane determined by P and Q is parallel to the plane determined by $-P$ and $-Q$. Hence, R and $-R$ lie in parallel planes. Moreover, the angle between $+R$ and $+P$ equals the angle between $-R$ and $-P$. Therefore, there results a couple, which we shall call the resultant couple.

64. The Axis of a Couple. — *A couple may be represented by a straight line.* Draw any line perpendicular to the plane of the couple, which we shall assume to be the plane of the paper. Lay off on this line a length proportional to the magnitude of the moment of the couple. Affix an arrowhead pointing toward the reader, if the tendency to rotate is counterclock-

wise; and in the opposite direction, if clockwise. This line is called the *axis* of the couple.

If we draw the axes of the P-couple, the Q-couple, and the R-couple of the preceding article, it is evident that they make the same angle with each other respectively that the planes of the couples do; and since the moments of the couples are respectively equal to $P \cdot \overline{AB}$, $Q \cdot \overline{AB}$, and $R \cdot \overline{AB}$, it follows that the length of the axes are proportional to P, Q, and R, respectively. Hence, the axis of the resultant couple R is the diagonal of a parallelogram of which the axes of the component couples P and Q are adjacent sides. Accordingly, we have the following

THEOREM. — *If the axes of two couples, not in parallel planes, are represented by the two adjacent sides of a parallelogram, the axis of the resultant couple is represented by the diagonal of the parallelogram passing through their intersection.*

It will be noted that this is the Parallelogram Law for the composition of forces, in which the word "force" is replaced by the words "axis of a couple"; and consequently, from each theorem concerning the resolution and composition of forces that is based directly on the Parallelogram Law, we may derive a corresponding thorem concerning couples, by replacing the word "force" by "axis of a couple."

For example, consider the particular case in which the axis of the P-couple, represented by OA, is perpendicular to the axis of the Q-couple, represented by OB; then the axis of the resultant couple is OC.

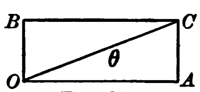


FIG. 31 *a*

Now $\qquad OA = OC \cos \theta$

and $\qquad OB = OC \sin \theta.$

Hence, we can resolve couples just as we do forces.

$$(\text{moment of } R)^2 = (\text{moment of } P)^2 + (\text{moment of } Q)^2.$$

The student will readily recall other theorems.

65. *We may combine a couple and a force acting in the same plane or parallel planes into a single force.*

Let the force be P and the forces of the couple be Q. Transfer the couple so that one of its forces acts in the same line and opposite to that of P (Art. 62). The resultant of $P - Q$ and Q is P, acting parallel to the original force at a point C, such that

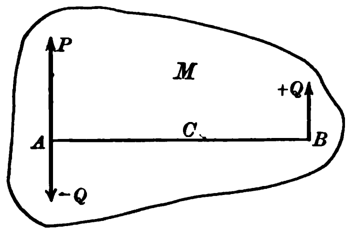


FIG. 32

$$(P - Q)\overline{AC} = Q \cdot \overline{BC}.$$

$$P \cdot \overline{AC} = Q(\overline{BC} + \overline{AC}) = Q \cdot \overline{AB} = \text{moment of couple.}$$

$$\therefore\ \overline{AC} = \frac{\text{moment of couple}}{P}.$$

Therefore, *the resultant of a couple and a force parallel to the plane of the couple is a force equal in magnitude and parallel to the original force, and at a distance from it equal to the moment of the couple divided by the magnitude of the force.*

NOTE. — Placing C between A and B is equivalent to assuming P greater than Q (Art. 50). The student will find it helpful to prove the proposition when P is less than Q.

66. We have shown that the resultant of any number of forces acting on a particle is a force; the resultant of any number of couples acting on a rigid body (Art. 63) is a couple; the resultant of a force and a couple acting on a rigid body is a force. We shall now prove the theorem.

THEOREM. — *Any number of forces acting on a rigid body is equivalent to a force and a couple.*

The proof that follows is for coplanar forces only. Choose as rectangular axes any lines OX and OY in the plane of the forces. Let P_1, P_2, $\cdots$ be a system of forces acting at x_1, y_1, x_2, y_2, $\cdots$, their lines of action making with the x-axis the angles θ_1, θ_2, $\cdots$.

Let the perpendicular distance of O from the lines of action of $P_1, P_2, \cdots$ be $p_1, p_2, \cdots$ respectively. Consider P_1. At O introduce two equal and opposite forces $+P_1$, $-P_1$; then, P_1, acting at x_1y_1, is equivalent to a couple ($+P_1$ acting at x_1y_1 and $-P_1$ acting at O) whose moment $= P_1p_1$ and to a force P_1 acting at O. Treat the other forces similarly. There results a system of forces P_1, P_2, $\cdots$ acting at O, and a system of couples whose moments are P_1p_1, $P_2p_2, \cdots$. Now the system of forces acting at O is equivalent to a force R, where, using the notation of Art. 26,

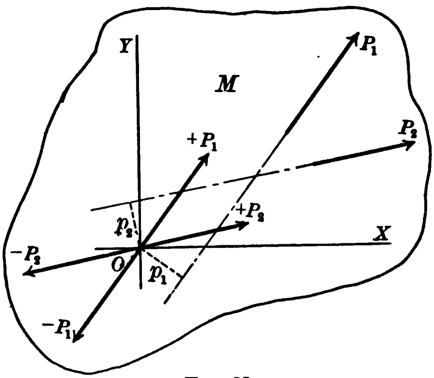


FIG. 33

$$R = \sqrt{X^2 + Y^2}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

R makes an angle θ with the x-axis such that

$$\tan\theta = \frac{Y}{X}; \quad \sin\theta = \frac{Y}{R}. \quad . \quad . \quad . \quad . \quad (8)$$

The couples $C_1 = P_1p_1$, $C_2 = P_2p_2 \cdots$ are equivalent to a single couple (Art. 63), whose moment

$$C = \Sigma C_i = \Sigma P_ip_i. \quad . \quad . \quad . \quad . \quad . \quad \text{Q.E.D.}$$

REMARK. — It will be noted that the force P_i has been replaced by another force equal and parallel to the original force and of the same magnitude. The resultant force is therefore independent of choice of origin or axes of coördinates (Art. 28). The resultant couple C, however, depends upon the choice of O, since this affects the value of $p_1, p_2, \cdots$, but not on the direction of the axes of coördinates.

By Art. 65, if $R \neq O$, the force R and the couple C are equivalent to a single force, which is the resultant of the system.

We do not often care to find it. But if we do, we compute R and C as above and apply Art. 65. That is, we move the line of action of R parallel to itself through a distance

$$p=\frac{C}{R}.$$

If $R = O$, the system is equivalent to a single couple, and there is no resultant; *that is, we have proved that, with one exception, every system of coplanar forces has one and only one resultant force.*

67. A System Equivalent to Any Number of Forces acting on a Rigid Body.

We shall consider two cases:

Case I. When the forces are coplanar.

Case II. When the forces are not coplanar.

It will be noted that Case I is a proof, in another way, of the theorem of Art. 66.

Case I. Let P_1, P_2, ⋯ be a system of coplanar forces acting on a rigid body, the coördinates of the points of application being x_1, y_1; x_2, y_2; ⋯, respectively.

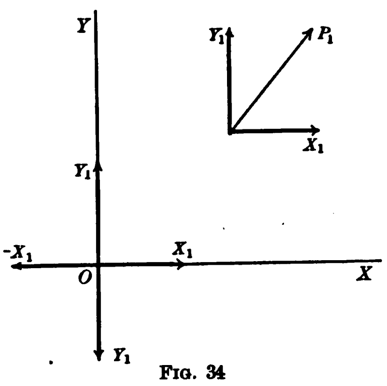


FIG. 34

Consider P_1. Resolve it into its components X_1, Y_1. Insert at O two forces X_1 and $-X_1$, acting along the x-axis, and two forces Y_1 and $-Y_1$ acting along the y-axis. The mechanical condition of the body has been unchanged. The force X_1 at x_1, y_1 and $-X_1$ at O constitute a couple whose moment is X_1y_1, and Y_1 at x_1y_1 together with $-Y_1$ at O constitute a couple whose moment is Y_1x_1. Since these two couples are in the same plane, the moment of the resulting couple is

$$C_1 = Y_1x_1 - X_1y_1.$$

There remain two forces, X_1, acting at O along the x-axis, and Y_1, acting at O along the y-axis.

We may treat each force as we have P_1. There will result (Art. 63) a single couple,

$$C = \Sigma(Y_i x_i - X_i y_i),$$

and a single force R, where

$$R = \sqrt{(\Sigma X_i)^2 + (\Sigma Y_i)^2}.$$

Case II. Let us suppose a rigid body is acted upon by a system of forces $P_1, P_2, \cdots$ which are not coplanar. Let the points of application of $P_1, P_2, \cdots$ be x_1, y_1, z_1; x_2 y_2, z_2; $\cdots$, respectively. Consider the force P_1. Replace it by its components X_1, Y_1, Z_1, acting at x_1, y_1, z_1.

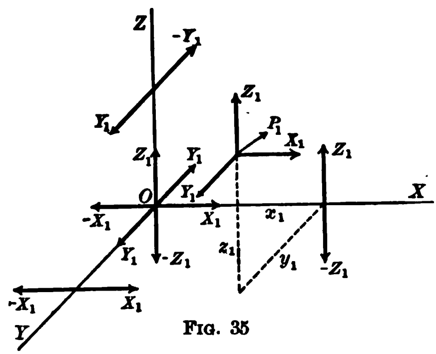


FIG. 35

Let us now insert

$Z_1, - Z_1$ at $x_1, 0, 0$,

$X_1, - X_1$ at $0, y_1, 0$,

$Y_1, - Y_1$ at $0, 0, z_1$,

and $X_1 - X_1$, $Y_1 - Y_1$, $Z_1 - Z_1$ at $0, 0, 0$.

The mechanical condition of the body is unchanged.

We may now combine these forces, so that there result three forces, X_1, Y_1, Z_1 acting at O and three couples:

$C_{1x} = Y_1 z_1 - Z_1 y_1$, a couple tending to rotate the body about the x-axis.

$C_{1y} = Z_1 x_1 - X_1 z_1$, a couple tending to rotate the body about the y-axis.

$C_{1z} = X_1 y_1 - Y_1 x_1$, a couple tending to rotate the body about the z-axis.

We have taken the moment of a couple as positive if, to an observer looking toward the origin from the positive end of the coördinate axis, the tendency is to rotate in a counter-

clockwise direction. We may treat each of the forces P_i as we have treated P_1, and for each one there result three forces, acting at the origin, equal respectively to X_i, Y_i, Z_i, the components of the force, and three couples tending to turn the body about the axes of coördinates.

We may find the resultant, R, of the forces acting at the origin, and, using the notation of Art. 29,

$$R = \sqrt{(\Sigma X_i)^2 + (\Sigma Y_i)^2 + (\Sigma Z_i)^2}.$$

Also the sum of the moments of the couples C_x, C_y, C_z about the x-, y-, and z-axes are, respectively:

$$C_x = \Sigma(Y_i z_i - Z_i y_i),$$
$$C_y = \Sigma(Z_i x_i - X_i z_i),$$
$$C_z = \Sigma(X_i y_i - Y_i x_i).$$

It is possible to find a single couple that will produce the same result as the three couples. Let C be the moment of this resultant couple. The student may prove, using Art. 64:

$$(a)\quad C = \sqrt{C_x^2 + C_y^2 + C_z^2}.$$

(b) If the axis of the couple C makes with the x-, y-, and z-axes, respectively, the angles α, β, γ, then

$$C_x = C \cos \alpha; \quad C_y = C \cos \beta; \quad C_z = C \cos \gamma.$$

CHAPTER IV*

VECTORS

68. Quantities used in mechanics may be divided into two classes, the basis of classification being that some of them, such as force, the axis of a couple, velocity, acceleration, etc., involve both magnitude and direction, while others, such as work, energy, the length of a line, etc., involve magnitude only.

A directed magnitude is called a *vector*. (See, however, Art. 71.) A magnitude which does not involve direction is called a *scalar*

69. Representation of Vectors. — A vector may be represented by the segment of a straight line affected with an arrowhead to indicate the direction or sense along the line. In Fig. 36 the vector AB is represented by the segment of the line *from* the *initial* point A to the *terminal* point B, the arrowhead signifying the sense or direction *along* the line. We shall designate the vector AB by $\underline{AB}$, which is read "the vector AB." If the length of AB is the scalar s, then s indicates the length of AB, while $\underline{s}$ means the vector represented by AB and is read "the vector s." Strictly speaking, the line AB is not a vector but represents a vector, a physical quantity of some kind, such as force or velocity. We shall however refer to the line AB when affected with an arrowhead as a vector. It must be noted that the vector AB is not the same as the vector BA, but that $\underline{BA}$ is the negative of $\underline{AB}$.

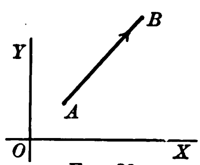


FIG. 36

* The most effective student of elementary mechanics is he who, at least in the beginning, lives closely to the physical interpretation of forces and their

70. Two vectors are equivalent if they have the same magnitude, direction, and sense. That is, if the line AB is parallel to CD and equal to it in length, and each has an arrowhead indicating the same direction, then $\underline{AB}$ is equivalent to $\underline{CD}$.

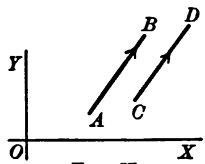


FIG. 37

71. The Addition of Vectors. — Historically, in the theory of vectors, as developed by Sir William Hamilton, the vector meant a step, and to add two vectors meant to take two successive steps. For example, one may move from A to C by taking a step from A to C, or by taking a step from A to B and then from B to C. The two processes produce the same final result.

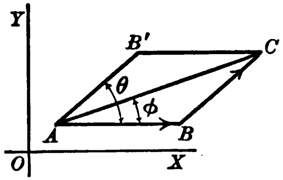


FIG. 38

This is expressed by the equation,

$$\underline{AC} = \underline{AB} + \underline{BC},$$

where the sign "=" means "produces the same result as" and the sign "+" means an additional process of the same kind. This process is called the addition of vectors, and $\underline{AC}$ is said to be the sum of $\underline{AB}$ and $\underline{BC}$.

Similarly, $\quad \underline{AC} = \underline{AB'} + \underline{B'C}.$

Now, since $\quad \underline{B'C} = \underline{AB}$, and $\underline{BC} = \underline{AB'}$,

it follows that $\quad \underline{AB} + \underline{BC} = \underline{BC} + \underline{AB}$;

effects. For this reason we have developed the Conditions of Equilibrium along the lines of the older methods. Moreover, it is necessary to show that forces, couples, etc., are compounded according to the Parallelogram Law before one can say that they are vectors. However, having developed the physical idea of a force, the student may, as he advances, declare his independence more and more of the elementary methods. For this reason we have introduced here as many of the essential features of the vector as the student can use with profit in the problems and theory that properly fall within the limits of this work.

that is, in the language of algebra, vectors obey the commutative law of addition.

We shall now define as vectors all directed magnitudes that can be added according to the Parallelogram Law.

We may also add two vectors $\underline{AB}$ and $\underline{CD}$, as follows:

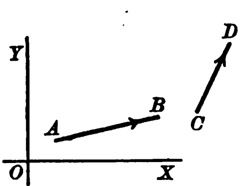


FIG. 39

Move $\underline{CD}$ parallel to itself, so that the initial point C, of $\underline{CD}$, coincides with the terminal point B of $\underline{AB}$, and D takes the position D'. Join A and D'. Then

$$\underline{AD'} = \underline{AB} + \underline{CD}.$$

COROLLARY. *It is evident that the scalar* (i.e. *the magnitude*) *of* AC (Fig. 38) *is given by the equation:*

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\,\overline{AB}\cdot\overline{BC}\cos\theta;$$

while
$$\frac{\overline{AC}}{\sin\theta} = \frac{BC}{\sin\phi},$$

give the direction that $\underline{AC}$ makes with $\underline{AB}$.

72. Subtraction of Vectors. — To subtract one quantity from another means merely that we add the negative of the subtrahend to the minuend. Consider again the vectors $\underline{AB}$ and $\underline{CD}$. $\underline{AB} - \underline{CD} = \underline{AB} + \underline{DC}$. Hence, by Art. 71, we move $\underline{DC}$ parallel to itself until D coincides with B, and C takes the position C_1.

Hence,
$$\underline{AB} - \underline{CD} = \underline{AC_1} = \underline{B'B}.$$

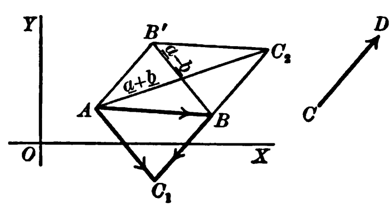


FIG. 40

It will be noted that if $\underline{AB} = \underline{a}$, and $\underline{CD} = \underline{b}$, the diagonal $\underline{AC_2}$ of the parallelogram $ABC_2B' = \underline{a} + \underline{b}$; while the diagonal $\underline{B'B}$ of the same parallelogram $= \underline{a} - \underline{b}$.

73. We add two equal vectors, that is, two parallel vectors with the same magnitude, by placing the initial point of one at the terminal point of the other. There thus results another vector, having the same direction but double the magnitude of either vector. We may add in the same way, m equal vectors. There would result a vector having the same direction as one of the original vectors, but with a magnitude m times this vector, m being any rational number or scalar. Hence, the product of a scalar m and a vector $\underline{s}$ is a vector $\underline{ms}$, having the direction of the original vector and a magnitude m times the vector $\underline{s}$.

74. Composition and Resolution of Vectors. — By a repetition of the process of Art. 71, it is evident that we may find the sum of any number of vectors. The sum of any number of vectors is called their resultant, and each separate vector is called a component of the resultant. It is also evident that the vectors need not all be in one plane. The process of finding the resultant of any number of vectors is called the composition of vectors; that of finding the components of a vector, the resolution of vectors.

75. There is one case of resolution of vectors of especial importance, viz.: that in which the vector $OC = \underline{F}$ is resolved into components $\underline{OA}$ and $\underline{OB}$ at right angles to each other. The components are then called the resolved parts of the vector, along the respective lines. From the figure it follows that the resolved part $\underline{OA}$ of $\underline{OC}$ is the projection of $\underline{OC}$ on $\overline{OA}$, and that

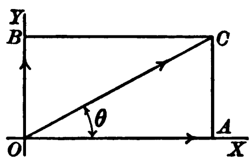


Fig. 41

$$\underline{OA} = \underline{OC} \cos\theta = \underline{F} \cos\theta.$$

Similarly,

$$\underline{OB} = \underline{OC} \sin\theta = \underline{F} \sin\theta.$$

$$\therefore\ OC = \sqrt{OA^2 + OB^2}.$$

76. It is desirable sometimes to resolve a vector along three lines at right angles to each other. Let the vector be resolved into the vectors along OX, OY, and OZ, respectively.

Let the angle that $\underline{F}$ makes with these lines be α, β, γ, respectively. Then $\underline{F}$ may be replaced by $\underline{OM}$ and $\underline{MP}$. $\underline{OM}$ may be replaced by $\underline{OL}$ and $\underline{LM}$.

Q.E.D.

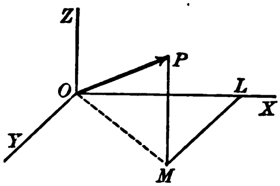


FIG. 42

Moreover,

$$\underline{OL} = \underline{F} \cos \alpha,$$
$$\underline{LM} = \underline{F} \cos \beta,$$
$$\underline{MP} = \underline{F} \cos \gamma.$$

$\underline{OL}$, $\underline{LM}$, and $\underline{MP}$ are called the resolved parts of the vector F along the axes OX, OY, and OZ, respectively.

77. The sum of the components along any line of any number of vectors is the sum of the projections of the vectors on that line. Let OA, AB, BC, and CD represent respectively $\underline{F_1}$, $\underline{F_2}$, $\underline{F_3}$, $\underline{F_4}$, making angles θ_1, θ_2, θ_3, θ_4, respectively, with OX.

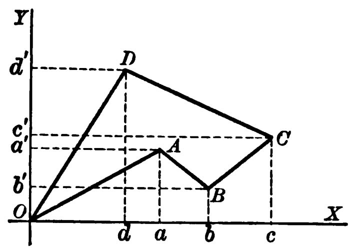


FIG. 43

Then the sum of the x-components of $\underline{F_1}$, $\underline{F_2}$, $\underline{F_3}$, $\underline{F_4}$ is $oa + ab + bc + cd = \underline{F_1} \cos \theta_1 + \underline{F_2} \cos \theta_2 + \underline{F_3} \cos \theta_3 + \underline{F_4} \cos \theta_4$; and the sum of the resolved parts of those vectors along OY is $oa' + a'b' + b'c' + c'd' = \underline{F_1} \sin \theta_1 + \underline{F_2} \sin \theta_2 + \underline{F_3} \sin \theta_3 + \underline{F_4} \sin \theta_4$. Moreover, the resultant of those vectors is $\underline{OD} = \underline{R}$ (say). If $\underline{R}$ makes an angle θ with the x-axis, then

$$\underline{R} \cos \theta = Od = \Sigma \underline{F_i} \cos \theta_i,$$
$$\underline{R} \sin \theta = Od' = \Sigma \underline{F_i} \sin \theta_i.$$
$$\therefore \tan \theta = \frac{\Sigma F_i \sin \theta_i}{\Sigma F_i \cos \theta_i}.$$

78. We have shown (Art. 15) that a force may be represented by a line with an arrowhead to indicate the sense along the line of action of the force. Moreover, it is resolved into components according to the Parallelogram Law. Force is therefore a vector. However, if the force vector represents in addition to the magnitude and direction of the force also its line of action, then the vector passes through a point (the point of application of the force) and is therefore a localized vector.

The theorems regarding the composition and resolution of forces are therefore special cases of the foregoing theorems on vectors. For example, the theorem of the preceding article may thus be stated. If any number of forces F_1, F_2, F_3, F_4 act on a particle, the resolved part of the resultant along any line equals the sum of the resolved parts of the forces. This is the theorem of Art. 26. In fact, the theorem of the polygon of forces, the conditions of equilibrium of forces acting on a particle, and many others, are special cases of the theory of vectors.

A couple is a vector, for it can be represented by a line with an arrowhead (Art. 64); and we find the resultant of two couples according to the Parallelogram Law (Art. 64).

The theorems of Art. 62 may now be stated thus: A couple may be represented by any vector of proper sense and magnitude which is perpendicular to the plane of the couple, and since the length of the vector is proportional to its moment, we can replace any couple by another of equal moment. Hence a moment is also a vector. We shall have many other illustrations of this principle. In order to show that any physical quantity is a vector it is necessary to show that it is a directed magnitude whose physical effect is not changed if we replace it by two quantities which may be represented by adjacent sides of a parallelogram of which the original quantity is a diagonal passing through their intersection.

It must be remembered that we can combine two vectors only if they represent quantities of the same kind. For example, we cannot combine a vector representing a force with one representing a couple.

CHAPTER V

THE STATICS OF A RIGID BODY

79. Equilibrium of Coplanar Forces Acting on a Rigid Body. — We have shown (Art. 66) that any system of coplanar forces acting on a rigid body can be replaced by a single force R and a couple C, and (Art. 65) that a force R and a couple C are equivalent to a single force R parallel to the original force whose line of action is at a distance d from the original line of action, where

$$d = \frac{\text{Moment of the couple}}{\text{Magnitude of the force}} = \frac{C}{R}.$$

Now the only single force that can produce equilibrium is one whose magnitude $= 0$. Hence, a *necessary* condition for equilibrium is that $R = 0$.

If $R = 0$, there remains a single couple which can produce equilibrium only if its moment is zero. Now the moment of the couple C about the point O is the sum of the moments of the forces about O. Hence, using the notation of Art. 66, $C = \Sigma P_i p_i = 0$ is a necessary condition for equilibrium. Hence, the necessary and sufficient conditions for equilibrium are

$$R = 0, \quad \text{and} \quad C = 0.$$

But if $R = 0$, then, using the notation of Art. 26,

$$X = \Sigma X_i = 0,$$
$$Y = \Sigma Y_i = 0.$$

Hence, the very important

THEOREM. — *The necessary and sufficient conditions for equilibrium of a system of coplanar forces acting an a rigid body are*

$$\left.\begin{aligned} X &= \Sigma X_i &= 0 \\ Y &= \Sigma Y_i &= 0 \\ C &= \Sigma P_i p_i &= 0 \end{aligned}\right\} \quad \cdots\cdots \quad (18)$$

It follows from Art. 67 in the same way that the necessary and sufficient conditions of equilibrium of any system of non-coplanar forces acting on a rigid body are

$$\left.\begin{aligned}&\Sigma X_i = 0,\\&\Sigma Y_i = 0,\\&\Sigma Z_i = 0,\\&\Sigma(Y_i z_i - Z_i y_i) = 0,\\&\Sigma(Z_i x_i - X_i z_i) = 0,\\&\Sigma(X_i y_i - Y_i x_i) = 0.\end{aligned}\right\} \quad . \ . \ . \ . \quad (18')$$

80. An important corollary.

Cor. *If a rigid body is in equilibrium under three forces, these forces pass through a point, or are parallel.*

Let P, Q, and R be three forces, two of which, P and Q, are not parallel to each other. Let their points of application be a, b, respectively. Transfer the points of application of these forces to A, the point of intersection of their lines of action.

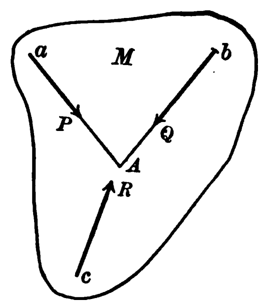


Fig. 44

Take moments about this point. The moments of P and Q with respect to A are zero. Therefore the sum of the moments of the forces with respect to this point equals the moment of R with respect to this point. And since the sum of the moments $= 0$, R must pass through A.

If P and Q are parallel (Art. 50), replace them by their resultant, which (Art. 57) is parallel to them and which to produce equilibrium with R must be equal and opposite to it and have the same line of action.

For example, let the rod AB, weighing Q pounds, rest against the smooth wall at A, the lower end B resting upon a rough surface at B. Let the pressure of the wall against the rod be represented by P. From the above discussion, it is evident

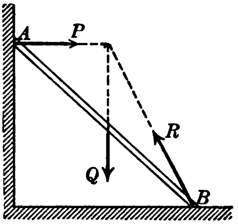


FIG. 45

that if the bar is in equilibrium, the line of action of the pressure R of the floor against the bar must pass through the point of intersection of P and Q. The student should note that when a rod is acted upon by three forces none of them act along the axis of the rod.

81. REMARK. — Equations (18) are the fundamental equations of statics. The student has observed that the equilibrium of a rigid body acted upon by coplanar forces affords *three and only three independent equations*. If, however, the forces are parallel, the number of independent equations is reduced to two, for the sum of the resolved parts perpendicular to the line of action of the forces is identically zero. If, further, the sum of the parallel forces is zero, there is only one independent equation.

One of these three equations will, in general, contain *linear* quantities (*i.e.* lengths of lines) either explicitly or implicitly, and usually the former, because a moment contains a linear quantity, viz., the arm of the moment. In general also, these equations will be homogeneous in each kind of quantities used. For example, if an equation contains a term in which a length enters to the first degree, a length of the first degree will enter into every term of the equation. If forces enter to the first degree in one term, forces will enter to the first degree in every term.

To solve problems, fix definitely in mind the rigid body whose equilibrium is being considered. *Make a sketch of the body, inserting every force acting on the body.* The student will not consider the magnitude of the force only, but must ascertain also its line of action; and he will make errors in this one particular more frequently than in any other. *Write the three independent equations of equilibrium.* If these equations contain more than three unknowns, either certain geometrical conditions will give additional equations, or the problem is impossible from the standpoint of statics alone.

The student will further note that since it is immaterial which lines he chooses as axes, he may, if he knows a body is in equilibrium, equate to zero the forces resolved along *any* line; and, since the moment of the forces around any point equals zero, he will also note that if he chooses a point through which some of the forces act as the origin of moments, he eliminates these forces from his equations.

82. A bar is said to be uniform if equal lengths of it weigh the same. We shall prove later that we may consider the

weight of a uniform bar as if it were a force acting at its middle point. For example, a uniform bar weighing 12 pounds is mechanically the same as if the bar had no weight, and is acted upon by a downward force of 12 pounds acting at its middle point.

83. The following problem illustrates some of the remarks of the preceding articles.

A smooth uniform bar, 5 feet long and weighing 10 pounds, rests on a vertical prop CD, $2\frac{1}{2}$ feet high; the lower end A rests on a smooth horizontal plane and is prevented from sliding by a string AD, $3\frac{1}{3}$ feet long. Find the tension of the string.

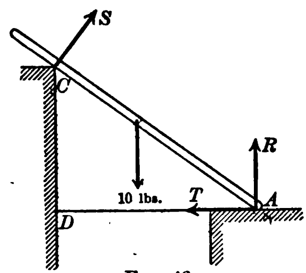


Fig. 46

In this case we consider the equilibrium of the bar. It is acted on by four forces: R, at right angles to the plane, for the plane is smooth; the weight of the bar; the tension of the string; and S, the reaction of the support at right angles to the bar, for the bar is smooth.

From the geometry of the figure, $AC = \dfrac{25}{6}$.

Taking moments about A, $(\Sigma P_i p_i = 0)$.

$$S\frac{25}{6} - 10 \cdot 2 = 0.$$

$$\therefore S = \frac{24}{5}.$$

Resolving horizontally, $(\Sigma X_i = 0)$.

$$S \cdot \sin DAC - T = 0,$$

$$\therefore T = S\frac{3}{5} = \frac{72}{25}.$$

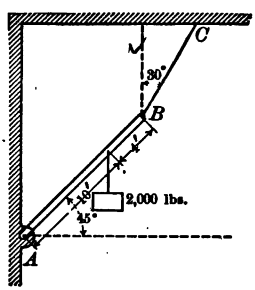
84. **Exercises**

1. A uniform bar 4 ft. long weighs 10 lbs.; weights of 30 lbs. and 40 lbs. are applied at its extremities. Where must it be supported to produce equilibrium?

Ans. 3 inches from middle of bar.

2. A man weighing 150 lbs. stands on a ladder 20 ft. long, resting with one end against a smooth vertical wall and the other end on a smooth horizontal plane. The ladder is prevented from slipping by a rope joining the foot of the ladder to the junction of the wall and the floor. What is the pressure on the wall, on the floor, and the tension of the rope when the man stands in the middle of the ladder and the foot of the ladder is 12 ft. from the wall?

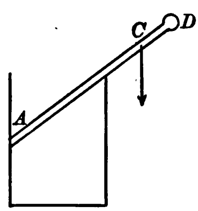
3. A uniform rod 12 ft. long weighing 25 lbs. rests with one end at the junction of a vertical wall and a horizontal floor; from a point 2 ft. from the other end a cord runs horizontally to a point 8 ft. above the floor. Find the tension of the cord and the pressure of the lower end of the rod.



4. Find the tension in the rope BC, and the amount and direction of the pressure at A, when the bar AB, loaded with a weight of 2000 pounds, 10 feet from A, makes an angle of 45 degrees with the horizontal.

5. A uniform bar 20 ft. long is supported by a smooth peg and the pressure of a smooth vertical wall upon one end of the bar. What will be the position of the bar for equilibrium, if the peg is 8 ft. from the wall?

6. A smooth glass rod AD rests in a smooth glass cylindrical tumbler, as shown in the figure. If the center of gravity of the rod is at C, 8 inches from A, and if the diameter of the tumbler is 3 inches, find the position of equilibrium of the rod.



85. The Method of Sections. — *Two-force Pieces.* In Arts. 42 and 45, we discussed frameworks made up entirely of members which were connected at their extremities by pins. In those articles it was pointed out that, if the weights of the members themselves were neglected, each member was subject to two forces only, and that in consequence, the direction of the stress in the member was along its axis. We shall call these two-force pieces. To make this clearer, consider, for example, the bar BC (Fig. 47). The line of action of the stress in BC must pass through the centers of the pins at B and at C. Moreover, the reaction of the pin at B on the bar BC must be equal and opposite to the reaction of the pin at C on BC. It follows, therefore, that the *internal forces*, those caused by the mutual interaction of the bodies of a system, are in equilibrium among themselves, and therefore the external or applied forces, such as the loads the truss carries and the reactions of the supports, must be in equilibrium among themselves. In Art. 45, the stresses were found by considering the equilibrium of each pin in turn. A second method of solution will now be presented.

If the members which make up a truss are accurately fitted and do not change form under the action of the loads, the entire framework or any part of it satisfies our definition of rigid body (Art. 47). We can, therefore, consider the equilibrium of the entire truss or any number of the members which make up the truss as a rigid body. Since the internal forces are in equilibrium among themselves, we need only consider those forces which are external to the part of the truss under consideration. Let us illustrate these points by analyzing the bridge truss in Fig. 47, in which all of the horizontal and vertical members are 10 feet in length and in which the loads

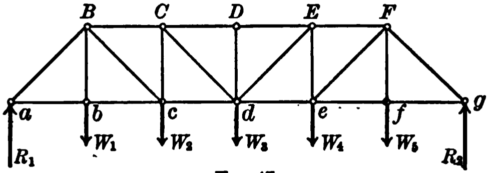


FIG 47

W_1, W_2, etc., are each 2000 pounds. Let it be required to find the stress in any member, Cd, for example.

We must first consider the entire truss as a rigid body and find the reactions at the supports a and g. Taking moments about g,

$$\Sigma P_i p_i = 0.$$

$$R_1 \cdot 60 - 2000(50 + 40 + 30 + 20 + 10) = 0.$$

$$\therefore R_1 = 5000 \text{ pounds.}$$

Resolving vertically, $(\Sigma Y_i = 0)$.

$$R_1 + R_2 - 10{,}000 = 0.$$

$$\therefore R_2 = 5000 \text{ pounds.}$$

Since the loading is symmetrical, we might have concluded that the reactions would have to be equal.

To find the stress in Cd, let us cut the truss in two, along the line 1–1 (Fig. 47 a), and introduce S_1, S_2, and S_3, acting along the members as indicated, and equal in magnitude to the stresses in them. Since the effect on the pins at C and c is unchanged, the part of the truss to the left of the section is a rigid body and is subject to exactly the same stresses that it was before the truss was cut. It is a rigid body in equilibrium under a system of non-concurrent coplanar forces. There are three conditions for equilibrium and three unknown forces. Therefore, a solution is possible.

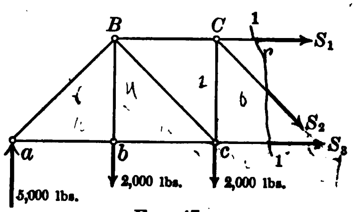


Fig. 47 a

To obtain S_2, the stress in Cd, let us resolve vertically,

$$(\Sigma Y_i = 0).$$

$$5000 - 4000 - S_2 \cos 45^\circ = 0.$$

$$\therefore S_2 = 1000\sqrt{2} \text{ pounds.}$$

We can easily find S_3, the stress in cd, by taking moments about C. Then

$$5000 \cdot 20 - 2000 \cdot 10 - S_3 \cdot 10 = 0.$$

$$\therefore S_3 = 8000 \text{ pounds.}$$

To find S_1, resolve horizontally, $(\Sigma X_i = 0)$.

$$S_3 + S_2 \cdot \cos 45° + S_1 = 0.$$

$$\therefore S_1 = -9000 \text{ pounds.}$$

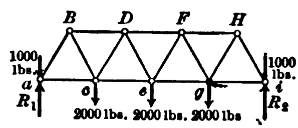
The positive sign of S_2 and S_3 shows that we chose them in the right direction; that is, these members are acting away from the pins and hence are in tension, and the negative sign of S_1 shows that we chose it in the wrong sense and that it is acting toward the pin at C and is therefore in compression. We will state the following rule:

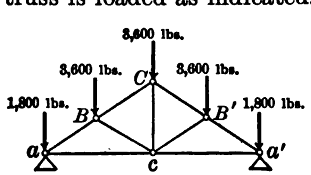
If, at the points of section, we introduce the forces as *acting away from the section*, and if in the equations of equilibrium *the known forces have the proper sign*, a *positive sign of a stress shows that it is in tension*, and a *negative sign that it is in compression.*

The method of sections is not applicable if the section cuts more than three unknown stresses. Why?

86. Exercises

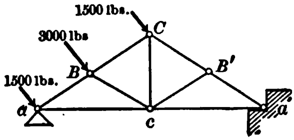
1. The figure represents a Warren bridge truss, the length of each member being 10 feet. The lower chord ai is horizontal. Find the stress in each member, if the truss is loaded as indicated.



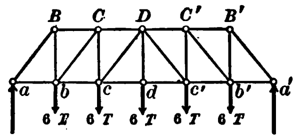


2. Find the stresses in the truss described in exercise 3, with vertical loads of 1800 pounds at a and a', and 3600 pounds at B, C, and B' as shown in this figure.

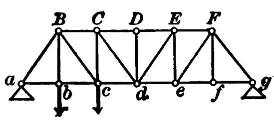
3. The length aa' of the truss shown is 30 feet; the length of Cc is 10 feet. $aB = BC = CB' = B'a'$. The end a' is supported in such a way that it can have both a vertical and a horizontal reaction, while the support at a is such that the reaction must be vertical. Find the stresses in each member when a load of 1500 pounds acts at a, 3000 pounds at B, and 1500 pounds at C, all of the loads acting normal to aC.



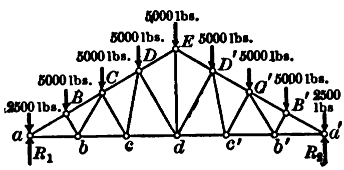
4. Each horizontal member of the Howe truss shown in the figure is 24 feet long; the depth of the truss is 30 feet. Find the stress in each member when the truss is under the loading shown in the figure.



5. Each horizontal member in the Pratt truss in the sketch is 25 feet long. The depth of the truss is 31 feet. Find the stress in each member of the truss due to loads of 58,000 pounds at b and at c.



6. Find the stresses in the Belgian roof truss shown in the figure. The members aB, BC, CD, DE, ED', $D'C'$, $C'B'$, and $B'a'$ are each 10 feet long; the angle at a is 30°; and Bb, Cc, Dd, and $D'd'$, $C'c'$, and $B'b'$ are perpendicular to aE and Ea', respectively. The member Ed is 20 feet long. The loads are as shown in the figure.



87. Three-force Pieces. — It is important that the student realize that the method of sections (Art. 85) can be applied to those members which are acted upon by two forces only. If any member is acted upon by three forces instead of two,

the lines of action of the forces are not along the axis of the member (Art. 80). It is true, however, just as in trusses, that the external forces are in equilibrium among themselves.

In finding the stresses in the members of such a structure, the student must first note which of the members can be cut. These will be recognized by the fact that they are acted upon by two forces only. Each piece that is acted upon by three forces must be treated just as the rod in Art. 83 is treated. Some members may be acted upon by four or more forces. These members must also be considered separately. The stresses in them can be found, if there are not more than three unknown forces.

When the direction of the reaction of a pin on a member is unknown, the student will find it convenient to represent it by two rectangular components, ordinarily chosen horizontally and vertically. In order that he may not confuse the components acting at the several points of the structure, the horizontal and vertical components of the force acting at the point A will be marked X_A and Y_A, respectively, and similarly for the other points B, C, ... He may direct these components as he pleases. If, however, he directs the known forces correctly, he will obtain a plus sign for the components correctly directed and a negative sign for those incorrectly directed. The negative sign, therefore, merely indicates that the direction of the component should be reversed.

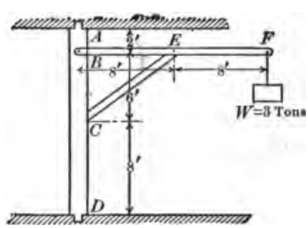


FIG. 48

These principles can be well illustrated by finding the reactions of the framework in Fig. 48.

Let it be required to find the pressures induced at A, B, C, D, and E by the load of 3 tons at F. The weights of the members themselves will be neglected. The cable which supports the weight W, and the inclined member CE are each acted upon by two forces and may be cut as in the method of sections.

The horizontal member *BEF* is acted upon by three forces, and the vertical member *AD* by four forces, so that each of these must be considered separately.

Let us first take the entire framework as a rigid body. The forces external to this body are the known weight W, equal to 3 tons, the unknown horizontal pressure X_A at A, the support being arranged so that no vertical component is possible. Also the unknown pressure at D, which we can replace by its horizontal component X_D and its vertical component Y_D.

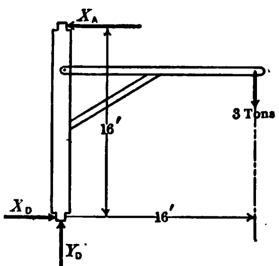


FIG. 48 *a*

As there is one known force and three unknown forces, the three conditions of equilibrium for a nonconcurrent system of coplanar forces make a solution possible.

Let us first take moments about D, thus eliminating X_D and Y_D.

$$16\,X_A - 6000 \times 16 = 0.$$

$$X_A = 6000 \text{ pounds}.$$

Summing the horizontal components, $(\Sigma X_i = 0)$.

$$X_D - X_A = 0.$$

Therefore, $$X_D = 6000 \text{ pounds}.$$

Summing the vertical components, $(\Sigma Y_i = 0)$.

$$Y_D = 6000 \text{ pounds}.$$

The amount of the pressure at D is therefore:

$$\sqrt{(X_D)^2 + (Y_D)^2} = \sqrt{(6000)^2 + (6000)^2} = 8484.6 \text{ pounds (nearly)}.$$

The direction of the pressure is given by

$$\theta = \tan^{-1}\left(\frac{Y_D}{X_D}\right) = \tan^{-1}(1) = 45^\circ.$$

Let us next consider the horizontal bar as a rigid body. The forces acting upon this body are, first, the known weight of

6000 pounds at F; second, an unknown force S in CE, which must act along the axis of the member, since it is a two-force piece; and finally, the unknown pressure of the pin at B against the bar, which is represented by the forces X_B and Y_B. The equations will be simplified if the center of moments is taken at B.

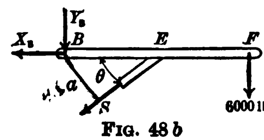


FIG. 48 *b*

Then, $$(\Sigma P_i p_i = 0).$$
$$6000 \cdot 16 + S \cdot a = 0.$$

From the geometry of the figure the lever arm a is evidently 4.8 feet.

Therefore, $$S = -\frac{6000 \cdot 16}{4.8} = -20{,}000 \text{ lbs.}$$

The negative sign indicates that the stress in CE should be reversed, just as in the method of sections; that is, that the stress is compression and not tension. To obtain X_B and Y_B, we have
$$-X_B - (-20{,}000 \cdot \cos\theta) = 0; \ (\Sigma X_i = 0).$$
$$\therefore\ X_B = 20{,}000 \times 0.8 = 16{,}000 \text{ pounds.}$$
$$-Y_B - 6000 - (-20{,}000 \sin\theta) = 0; \ (\Sigma Y_i = 0).$$
$$Y_B = 20{,}000 \cdot 0.6 - 6000 = 6000 \text{ pounds.}$$

The amount and the direction of the pressure at B are found in the same way as at D. The pressure is 17,100 pounds (nearly), making an angle of 200° (nearly) with the positive direction of the x-axis.

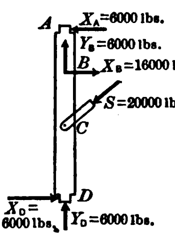


FIG. 48 *c*

All of the forces which act upon the vertical member have now been determined. The computations may be checked if we consider this member as a rigid body and show that it is in equilibrium if acted upon by those forces. The forces acting are shown in the sketch. The student should keep clearly in mind that he is now representing the action of all of the members on AD. The following equations, in

which we have substituted the values obtained above, check our solution.

$$X_D - X_A - S\cos\theta + X_B = 0;\ (\Sigma X_i = 0).$$
$$6000 - 6000 - 16{,}000 + 16{,}000 = 0.$$
$$Y_B - S\sin\theta + Y_D = 0;\ (\Sigma Y_i = 0).$$
$$6000 - 12{,}000 + 6000 = 0.$$
$$20{,}000 \cdot 8 \cdot 0.8 + 6000 \cdot 16 - 16{,}000 \cdot 14 = 0;\ (\Sigma P_i p_i = 0).$$
$$128{,}000 + 96{,}000 - 224{,}000 = 0.$$

In the simple framework which has just been solved, three rigid bodies have been considered. In each case, there were three unknown elements and three equations and a solution was possible. It is sometimes necessary to consider two parts of a framework before we can obtain the desired unknowns. This will be illustrated by the solution of the A-frame shown in Fig. 49. The student will note that this framework is composed entirely of members acted upon by three forces. Let it be required to find the amount and the direction of the pressure of the pin at B on the member ABC, of the pin at C on the member ABC, and of the pin at D on the member BD.

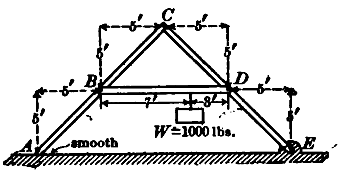


FIG. 49

Let us first consider the entire framework as a rigid body. Since the surface at A is smooth, the reaction at that point, Y_A, must be vertical. At E, we have the unknown pressure represented as usual by X_E and Y_E. Hence

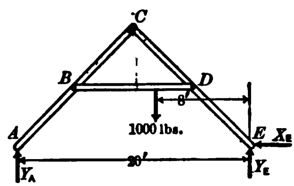


FIG. 49 *a*

$$1000 \cdot 8 - Y_A \cdot 20 = 0;\ (\Sigma(P_i p_i)_E = 0).$$
$$\therefore\ Y_A = 400 \text{ pounds}.$$
$$X_E + 0 = 0;\ (\Sigma X_i = 0).$$
$$\therefore\ X_E = 0.$$
$$Y_A - 1000 + Y_E = 0;\ (\Sigma Y_i = 0).$$
$$\therefore\ Y_E = 1000 - 400 = 600 \text{ pounds}.$$

Now that Y_A and Y_B are known, it is evident that there is one known force acting upon each of the three bars. Since the direction and amount of the pressures of the pins at B, C, and D are each unknown, it is immaterial which bar is chosen for consideration.

Let us take the member BD, representing the pressures at B and at D by X_B and Y_B and by X_D and Y_D, respectively. As there are four unknown components and only three conditions, it will not be possible to evaluate completely the pressures at B and D by the consideration of this rigid body alone. Let us choose B as our center of moments, thereby eliminating three of our unknowns.

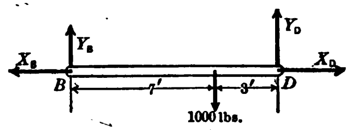


FIG. 49 *b*

$$Y_D \cdot 10 - 1000 \cdot 7 = 0; \ (\Sigma(P_i p_i)_B = 0).$$
$$\therefore \ Y_D = 700 \text{ pounds.}$$
$$Y_B + 700 - 1000 = 0; \ (\Sigma Y_i = 0).$$
$$\therefore \ Y_B = 300 \text{ pounds.}$$
$$X_B = X_D; \ (\Sigma X_i = 0).$$

Let us next consider the member ABC. At A, we have Y_A acting vertically upward and equaling 400 pounds. Since the pin at B passes through two bars only, the pressure of the pin against the bar ABC must be exactly equal and opposite to the pressure of the pin against the bar BD. Therefore, we have at B in this rigid body, two components X_B and Y_B, equal and opposite to those acting on BD. At C, we have two components, X_C and Y_C. As there are two known forces and only three unknown forces, a definite value for the pressures at B and C can be obtained. As soon as we have a value for X_B, we know X_D and our problem is solved.

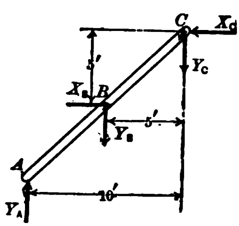


FIG. 49 *c*

$$X_B \cdot 5 + 300 \cdot 5 - 400 \cdot 10 = 0; \ (\Sigma(P_i p_i)_C = 0).$$

Therefore, $X_B = \dfrac{4000 - 1500}{5} = 500$ pounds.

$$X_B - X_C = 0; \; (\Sigma X_i = 0).$$
$$\therefore \; X_C = 500 \text{ pounds.}$$
$$400 - 300 - Y_C = 0; \; (\Sigma Y_i = 0).$$
$$\therefore \; Y_C = 100 \text{ pounds.}$$

The pressure of the pin at B against the bar ABC is 583.1 pounds (nearly), making an angle of 329° 02′ with the positive direction of the x-axis. Similarly, the pressure of the pin at C against the bar ABC equals 509.9 pounds, making 191° 19′ with the x-axis, and the pressure of the pin at D against the bar BD is 860.2 pounds making an angle of 54° 28′ with the x-axis.

In each of the above solutions we have made use of the fact that when two members act on a pin, the action of the pin on one member is exactly equal and opposite to the action on the second member. When more than two members act upon a pin, this is not true. To illustrate this, let us solve the crane in Fig. 50. As usual, we will neglect the weights of the members. The cable, the member CE, and the tie rod AH are each acted upon by two forces and hence can be cut.

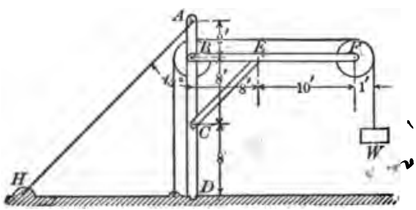


Fig. 50

The horizontal member BEF is acted upon by forces at B, at E, and at F; and the vertical member receives pressures at A, B, C, and D, so that each of these members must be considered separately.

Let the weight, $W = 4000$ pounds, be given, and let it be required to find the tension or compression in each two-force piece, and the pressures of the pin at B upon each member through which it passes.

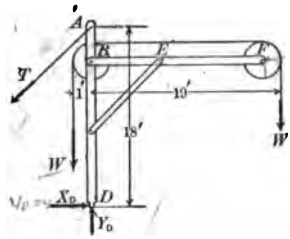


FIG. 50 *a*

Let us first take the entire framework as a rigid body. The forces external to this rigid body are the weight, $W = 4000$ pounds, the pull in the cable, also equal to 4000 pounds (Art. 42), the unknown tension T in the member AH, and a pressure at D unknown in amount and direction. Then

$$T \cdot 18 \sin 45° + W \cdot 1 - W \cdot 19 = 0; \quad (\Sigma(P_i p_i)_D = 0).$$

$$\therefore T = \frac{18 \cdot 4000}{18 \cdot 0.707} = 5657.7 \text{ pounds (nearly)}.$$

$$\Sigma X_i = 0 \text{ and } \Sigma Y_i = 0 \text{ give } X_D = 4000 \text{ pounds},$$

and $\quad Y_D = 12{,}000$ pounds.

Let us consider the member BEF. At F, we have the pressure brought by the pulley; at E, the unknown stress S in CE; and at B, the unknown pressure of the pin at B against the bar BF, represented by X'_B and Y'_B. Therefore

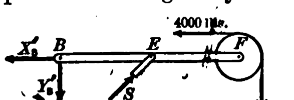


FIG. 50 *b*

$$4000 \cdot 1 - 4000 \cdot 19 + S \cdot 8 \cdot \sin 45° = 0; \quad (\Sigma(P_i p_i)_B = 0).$$

$$\therefore S = \frac{18 \cdot 4000}{8 \cdot 0.707} = 12{,}730 \text{ pounds (nearly)}.$$

$$S \sin 45° - 4000 - X'_B = 0; \quad (\Sigma X_i = 0).$$

$$\therefore X'_B = -4000 + 9000 = 5000 \text{ pounds}.$$

$$S \cos 45° - 4000 - Y'_B = 0; \quad (\Sigma Y_i = 0).$$

$$Y'_B = 9000 - 4000 = 5000 \text{ pounds}.$$

The amount of the pressure of the pin at B on the bar BF is 7071 pounds (nearly) and its direction is 225° with the positive direction of the x-axis.

Finally, consider the vertical member as a rigid body. We now know T, S, X_D, and Y_D as shown in the figure. There

remains the unknown pressure of the pin at B against this member. Let us represent this reaction by the components X''_B and Y''_B. Since the pulley also acts upon the pin at B, X''_B and Y''_B are *not* equal and opposite to X'_B and Y'_B, respectively. Hence,

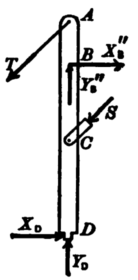


FIG. 50 c

$$X''_B - T\cos 45° - S\cos 45° + X_D = 0; \quad (\Sigma X_i = 0).$$
$$X''_B = 4000 + 9000 - 4000 = 9000 \text{ pounds}.$$
$$Y_D + Y''_B - T\sin 45° - S\sin 45° = 0; \quad (\Sigma Y_i = 0).$$
$$Y''_B = 4000 + 9000 - 12{,}000 = 1000 \text{ pounds}.$$

The amount of the pressure of the pin at B against the member AD is, therefore, 9050 pounds (nearly) and its direction 6° 20′ (nearly) with the positive direction of the x-axis.

We have still to find the amount and direction of the pressure of the pin at B against the pulley. Using the pulley as a rigid body under forces as shown in the sketch, we find

$$X'''_B = 4000 \text{ pounds}; \quad (\Sigma X_i = 0).$$
$$Y'''_B = 4000 \text{ pounds}; \quad (\Sigma Y_i = 0).$$

The amount of the pressure is therefore 5650 pounds (nearly), and the direction 135°.

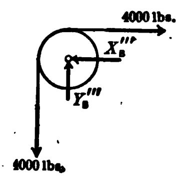


FIG. 50 *d*

88. Exercises

1. Four forces of 5, 3, 8, and -6 pounds act perpendicularly to a rigid rod at the points P, Q, R, and S, such that $PQ = 5$ feet; $QR = 4$ feet; and $RS = 3$ feet. Find the magnitude and the point of application G of the resultant force.

Ans. $R = 10$ pounds; $PG = 1.5$ feet.

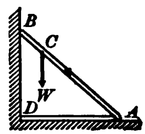
2. A uniform bar AB is 10 feet long and weighs 50 pounds; from the extremity A is suspended a weight of 5 pounds and from the extremity B a weight of 1 pound. At what point must the bar be supported in order that it be in equilibrium?

3. AB is a rigid weightless rod two feet long. W at A just balances on a knife-edge two pounds at B. When the fulcrum is shifted toward B two inches, W at A balances three pounds at B. Find W and the position of the fulcrum.

Ans. $W = 6$ pounds or 1 pound.

4. Find the true weight of a body which is found to weigh 8 ounces and 9 ounces when placed in each of the scalepans of a false balance. *Ans.* $6 \cdot \sqrt{2}$ ounces.

5. A weightless rod AB rests with one end A on a smooth horizontal plane. The other end B against a smooth vertical wall BD. A weight W is suspended from C, where $AC = a$. The rod is prevented from slipping by a string AD. Find the reaction at A, at B, and the tension of the string. Let the length of the beam $= l$, and the angle that the rod makes with the horizontal $= \alpha$.

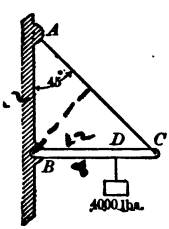


Ans. Reaction at $A = W$; reaction at $B = T = \frac{a}{l} \cdot W. \cot \alpha$.

6. If instead of the smooth horizontal floor and string the end A were prevented from slipping by a hinge, find the reaction of the hinge.

Ans. Reaction at $A = \frac{W}{l}\sqrt{l^2 + a^2 \cot^2 \alpha}$; $\tan \theta = \frac{l}{a} \tan \alpha$,

where θ is the angle that the reaction makes with the horizontal.



7. The bar BC is hinged at B and is held in a horizontal position by the cable AC as in the figure. This bar is 12 feet long and supports a weight of 2 tons at D, the length BD being 8 feet. If the weight of BC is 400 pounds, applied midway between B and C, find the tension in the cable AC, and the amount and direction of the pressure of the pin at B on the bar BC.

8. A uniform smooth rod $AB = 8$ feet, weighing 20 pounds, rests with one end A on a smooth horizontal plane AD, and with the point E on a support whose height above AD is $DE = 3$ feet. A horizontal cord $AD = 4$ feet holds the rod in equilibrium. Find the tension T of this cord, and the reactions at A and E.

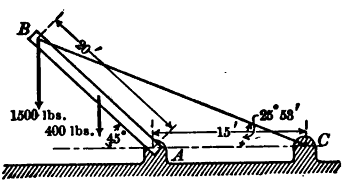
Ans. $T = 7.68$ pounds; the reaction at $E = 12.80$ pounds; the reaction at $A = 9.76$ pounds.

9. A uniform rod, $AB = 2l$, weighing W pounds, rests with its upper end A against a smooth vertical wall. The other end B is supported by a string $BC = 2a$, attached to a point C in the wall, the rod and the string being in a plane perpendicular to the wall. Find the position of equilibrium, the tension of the string, and the reaction at A. Note the body is in equilibrium under three forces, which pass through a point (Art. 78).

Ans. Let $\angle ACB = \phi$.

Then, $$\tan\phi = \frac{1}{2}\sqrt{\frac{4l^2 - a^2}{a^2 - l^2}}.$$

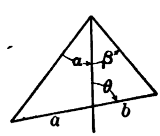
$T = W\sec\phi$; reaction at $A = W\tan\phi$.*



10. AB is a stiff uniform rod, hinged at A, so as to move freely in the vertical plane AB. BC is a string. With the dimensions as shown in the figure, find the tension of the string and the horizontal and vertical reactions at A.

Ans. $T = 3670$ pounds (nearly); vertical pressure $= 3502$ pounds (nearly).

* A theorem from trigonometry which may frequently be applied to the mechanics of a rigid body in equilibrium under three forces is: Given a line drawn through the vertex of a triangle as shown, then



$$(a + b)\cot\theta = a\cot\alpha - b\cot\beta.$$

11. Two equal smooth cylinders rest in contact on two smooth planes inclined at angles α and β to the horizon; find the inclination θ to the horizon of the line joining their centers.

Ans. $\tan\theta = \frac{1}{2}(\cot\alpha - \cot\beta)$.

Hint. — See Note, exercise 9.

12. A carriage wheel whose weight is W and whose radius is r rests on a level road. Show that any horizontal force acting through the center of the wheel greater than

$$P = W\frac{\sqrt{2rh - h^2}}{r - h}$$

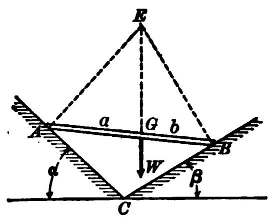
will pull it over an obstacle whose height is h.

13. A weightless rod AB rests on two smooth planes inclined to the horizon at angles α and β, and carries a weight W at the point G. The intersection C of these planes is horizontal and at right angles to the vertical plane through AB. Find the inclination θ of AB to the horizon, and the pressures at A and B.

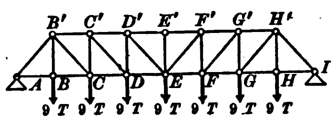
Ans. $\text{Tan}\,\theta = \dfrac{a\cot\alpha - b\cot\beta}{a + b}$.

$$\text{Reaction at } B = \frac{\sin\alpha}{\sin(\alpha+\beta)}W.$$

$$\text{Reaction at } A = \frac{\sin\beta}{\sin(\alpha+\beta)}W.$$



14. Let the figure represent a Pratt truss of eight equal panels. Let us assume that the truss weighs 9 tons per panel, and that this weight is concentrated at the lower joints as shown. Find the stresses in the bars.

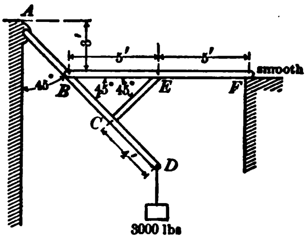


This weight of the truss is usually called the "dead load."

Let $AB = BC = 22$ feet, and $BB' = 26$ feet.

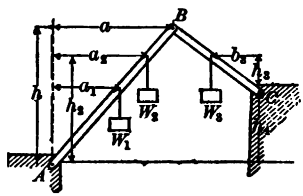
Ans. Stress in $EF' = 5.89$ tons, tension; in $DE = 57.1$ tons, tension; in $C'D' = 57.1$ tons, compression.

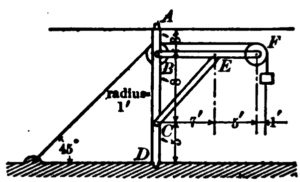
15. Using the figure in exercise 14, suppose that in addition to the "dead load," locomotive wheels rest on the truss so as to cause an additional load of 50 tons at G and 50 tons at H. Find the stress in EF.



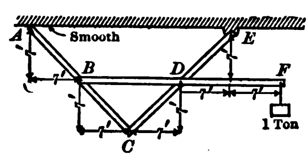
16. Three bars, whose lengths and angles with the horizontal are shown in the figure, support a load of 3000 pounds. Find the pressure on the smooth surface F, and the pressure of the pins at A, B, C, and E on each member through which they pass. Neglect the weight of the bars.

17. The two links AB and BC, considered weightless, support the weights W_1, W_2, and W_3 as shown. If the length of AB is l_1 and that of BC is l_2, and if the angles that AB and BC make with the horizontal are α and β, respectively, and if the distances to the points of application of the loads are as shown in the figure, find the reaction of the pins at A and B on the bar AB.



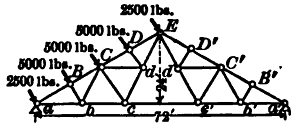


18. The crane in the figure is supporting a load of four tons as shown. Find the pressure of the pins at B and at C on each member through which they pass. All geometrical elements are as in the figure.



19. The framework in the sketch supports a load of one ton as shown. Find the pressures of the pins at B, C, and D against each of the members through which they pass.

20. The steel roof truss in the figure is acted upon by the wind load, normal to aE as shown. The span is 72 feet; the rise is 24 feet. Members Bb, Cc, Dd are perpendicular to aE; AB, BC, CD, and DE being equal. The right end of the truss rests on rollers to allow for changes in length due to temperature. Find the stresses in the members.



Hint. — Since the right end of the truss rests on rollers, the right reaction can have no horizontal component. To obtain the stress in the member cc' take the right half of the truss as a rigid body and sum the moments about E.

CHAPTER VI

THE CENTER OF GRAVITY

89. A special, but very important, case of the theory of parallel forces is that one in which the forces P_i (Art. 53) are replaced by the weights of particles. A body may be considered as being made up of a number of particles each attracted toward the center of the earth; and since the bodies that we shall consider are small compared with the radius of the earth, the weights of the particles act in sensibly parallel lines. The center of all these forces is called the center of gravity of the body.

From this it appears that the *weight of the body* which is the sum of the weights of its various particles, and therefore the resultant of all the weights, — *may be considered as acting at the center of gravity of the body.*

90. We shall speak of the center of gravity of lines and of surfaces. By a line in this sense we shall mean an infinitely thin rod made up of an infinite number of thin sections placed end to end, or side to side; and a surface may be built up by laying side by side an infinite number of thin rods; and a solid may be built up by laying upon each other an infinite number of thin laminæ. A homogeneous body is one in which each cubic unit has the same weight. All other bodies are heterogeneous. Unless explicitly stated we shall assume that all bodies with which we work are homogeneous, and in the case of areas that they are of uniform thickness.

91. Some Special Cases. — (*a*) *The center of gravity of a uniform rod is at its middle point.* For, let AB be such a rod; at A and B there are two particles of the same volume, therefore

of equal weight. The center of these parallel forces is at C, the middle of the rod. Considering in turn every pair of such particles, we see that the center of force is at the middle point.

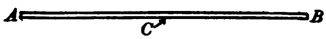


FIG. 51

(*b*) *The center of gravity of a homogeneous triangular area of uniform thickness is at the intersection of the medians.* Let ABC be such a triangle. Consider the triangle as made up of thin rods parallel to BC. The center of gravity of each of these rods is at its middle point; therefore, the weight may be considered as concentrated in the median through A; and therefore the center of gravity is somewhere in this line. We may also regard the triangle as made up of rods parallel to AC, and therefore the center of gravity of the area is somewhere on the median through B. Therefore, the center of gravity is at the intersection of the medians.

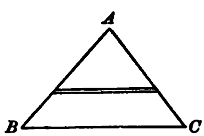


FIG. 52

(*c*) The student may prove that *the center of gravity of a parallelogram is at the intersection of the diagonals of the parallelogram.*

(*d*) *The center of gravity of a regular hexagon is at its center.*

92. The Principles of Symmetry often enable one to find easily the center of gravity of a body. A plane figure is mechanically symmetrical with respect to a given line (and a solid or surface with respect to a given plane), if to every particle on one side of the line (or plane) there is a corresponding particle of the same weight so situated that the line joining the two corresponding particles is perpendicular to the line (or plane) and bisected by it. It is evident from equations (15) and (16) that if for any figure we have two lines or three planes of symmetry, the center of gravity lies at their intersection.

For example, any diameter of a uniform circular area is a line of symmetry, and therefore the center of gravity of a uni-

form circular area is at its center. So also, either the major axis or minor axis of an elliptical area or elliptical ring is a line of symmetry, and therefore the center of gravity lies at the center.

93. The Center of Gravity of Compound Bodies. — It is advantageous many times to consider a body as made up of a finite number of bodies, each of whose centers of gravity we know.

Let the body M be made up of n bodies $M_1, M_2, \cdots M_n$, whose volumes are $V_1, V_2, \cdots V_n$ and whose weight per unit volume $= \rho_1, \rho_2, \cdots \rho_n$, respectively; then the weights of these bodies are $V_1\rho_1, V_2\rho_2, \cdots V_n\rho_n$. Let the coördinates of the center of gravity of the bodies $M_1, M_2, \cdots M_n$ be x_1, y_1, z_1; x_2, y_2, z_2; $\cdots x_n, y_n, z_n$, respectively. If $\bar{x}, \bar{y}, \bar{z}$ are the coördinates of the center of gravity of the body M, then by substituting in (15), (16), and (17) of Art. 53, we obtain

$$\bar{x} = \frac{V_1\rho_1 x_1 + V_2\rho_2 x_2 + \cdots V_n\rho_n x_n}{V_1\rho_1 + V_2\rho_2 + \cdots V_n\rho_n} = \frac{\Sigma_i V_i\rho_i x_i}{\Sigma_i V_i\rho_i}$$

and
$$\bar{y} = \frac{\Sigma_i V_i\rho_i y_i}{\Sigma_i V_i\rho_i};\ \bar{z} = \frac{\Sigma_i V_i\rho_i z_i}{\Sigma_i V_i\rho_i}.$$

If $\rho_1 = \rho_2 = \cdots \rho_n$, these equations may be written

$$\bar{x} = \frac{\Sigma_i V_i x_i}{\Sigma_i V_i};\ \bar{y} = \frac{\Sigma_i V_i y_i}{\Sigma_i V_i};\ \bar{z} = \frac{\Sigma_i V_i z_i}{\Sigma_i V_i}.$$

In many problems engineers find it necessary to find the center of gravity of cross sections of beams. Let the thickness of the cross section be t, and let the section be homogeneous. Suppose the cross section of area A be made up of the areas $A_1, A_2, \cdots A_n$, whose centers of gravity are at x_1, y_1; x_2, y_2; $\cdots x_n, y_n$, respectively. The volume $V_i = tA_i$. Hence, since t is constant,

$$\bar{x} = \frac{A_1x_1 + A_2x_2 + \cdots A_nx_n}{A_1 + A_2 + \cdots A_n} = \frac{\Sigma_i A_i x_i}{A}.$$

$$\bar{y} = \frac{\Sigma_i A_i y_i}{A}.$$

For example, the T-section $ABCDEFGH$ (Fig. 53) may be thought of as made up of two rectangles $ABCD$ and $EFGH$. We shall consider the section as homogeneous; the line OP being a line of symmetry of the figure, the center of gravity lies on this line.

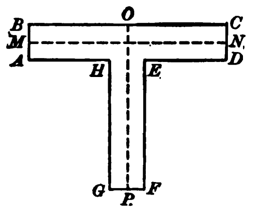


FIG. 53

Let $AB = 2a$; $GH = 2c$;
$BC = 2b$; $GF = 2d$.

Take O as origin; OC as x-axis.

If, now, $\bar{x}$, $\bar{y}$ be the coördinates of the center of gravity $\bar{x} = 0$;

the weight of $ABCD = k \cdot 4\,ab$,

and the weight of $EFGH = k \cdot 4\,cd$.

We may think of these weights as acting at points whose y-coördinates are respectively a and $2a + c$.

$$\therefore \bar{y} = \frac{ab \cdot a + cd(2a + c)}{ab + cd} = \frac{a^2b + 2\,acd + c^2d}{ab + cd}.$$

Another example: Consider an artist's palette as made up of a circular area of radius 6 inches in which is bored a hole of radius 1 inch, the center of the small circle being 4 inches from the center of the large one. Find the center of gravity of the palette. By symmetry it is on the line joining the centers of the circular areas. Let the centers be O and C, respectively. Take OC as the x-axis, and O as the origin, $\bar{x}$, $\bar{y}$ as the coördinates of the center of gravity. Then $\bar{y} = 0$.

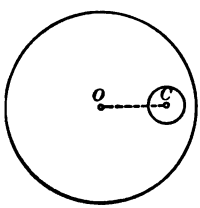


FIG. 54

The circle before the hole was bored may be considered as made up of the small circle and the palette.

The area of palette $= \pi(36 - 1)$ square inches.

The area of small circle $= \pi \cdot 1$ square inches.

$$\bar{x}_1 = \frac{\pi\,35 \cdot \bar{x} + \pi\,1 \cdot 4}{\pi\,35 + \pi \cdot 1}.$$

Since the center of gravity of the original circle is at the origin $\bar{x}_1 = 0$. We have, therefore,

$$\bar{x} = -\frac{4}{35} \text{ inch.}$$

A third example: To find the center of gravity of a trapezoid $ABCD$. Let $AB = a$, $CD = b$, and the altitude of the trapezoid $= h$. Evidently, G lies on the line joining the middle points of the parallel sides. Moreover, choosing AB as x-axis,

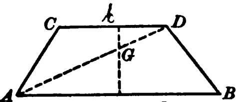


Fig. 55

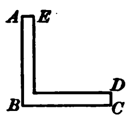
$$\bar{y} = \frac{\triangle ACD \cdot \frac{2h}{3} + \triangle ABD \cdot \frac{h}{3}}{\text{Area } ABCD}$$

$$= \frac{h}{3} \cdot \frac{2b + a}{a + b}.$$

The special methods indicated in Arts. 91, 92, and 93, when applicable, lead more directly to results than the more general method, to be developed presently, and should be employed when possible.

94. Exercises

1. Find the center of gravity of the section of an angle iron where



$$AE = 1 \text{ inch} \quad = CD$$

and $$AB = 6 \text{ inches} = BC.$$

2. Find a line of symmetry of a parabolic area; of a right circular cylinder; two lines of symmetry of a rectangle.

3. Find the center of gravity of a rectangle; a rectangular area; a cylinder; a cylindrical area; a sphere; a spherical shell.

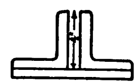
4. A rectangular area 8 inches long and 5 inches wide has cut from one end an isosceles triangle whose base coincides with one end of the rectangle and whose altitude equals 4 inches. Find the center of gravity of the remaining area.

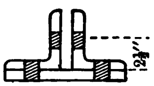
Ans. On line of symmetry $\frac{4}{9}$ inch from center of rectangle.

5. The blade of a T-square is 30 inches by $2\frac{1}{2}$ inches. A hole $\frac{1}{2}$ inch in diameter is bored through the blade, the center of the hole being 2 inches from the end. Find the center of gravity of the blade.

6. An equilateral triangular area is mounted on one side of a square area one of whose sides is the base of the triangle, and in the same plane. Find the center of gravity of the entire figure.

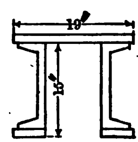
7. The plate in the sketch is $11'' \times \frac{9}{16}''$; the angles, $5'' \times 4'' \times \frac{9}{16}''$. If the area of the angles is 4.75 square inches and the distance to the center of gravity from the long leg 1.10 inches, find the center of gravity of the section.





8. How much nearer the plate would the center of gravity be if rivet holes 1'' wide were removed from the section as in the sketch?

9. The cross section of the end post of a bridge is made up of a plate $19'' \times \frac{3}{8}''$, two 15-inch channels, each with an area of 9.90 square inches, and two flats $4'' \times \frac{3}{4}''$. Find the distance to the center of gravity of the section from the outside of the 19-inch plate.



10. A mallet is made by boring a hole $1\frac{3}{4}''$ in diameter in the middle of a cylindrical block of wood, whose diameter and length are 6'' and 10'', respectively. How long should the handle be, that the center of gravity of the mallet may be in the handle, 2'' from the head of the mallet? The handle is $1\frac{3}{4}''$ in diameter.

11. A cast-iron rectangular parallelopipedical block, $2'' \times 8'' \times 5''$, weighs 150 pounds and its center of gravity is found to be 5 inches from the left end. What is the size of the blow hole, and how far is its center of gravity from the left end?

95. General Formulæ for Coördinates of Center of Gravity. — The coördinates of the center of gravity may be expressed in one general formulation, which will apply to bodies of any shape and any density whatever. From these general formulæ we can easily deduce expressions for all special cases.

Let a given body be divided in any way into small pieces, and let us denote the volume of these pieces by

$$\Delta V_1, \Delta V_2, \dots \Delta V_n,$$

their weights by

$$\Delta W_1, \Delta W_2, \dots \Delta W_n,$$

and their weight per cubic unit by

$$\rho_1, \rho_2, \dots \rho_n, \text{ respectively.}$$

Then,

$$\Delta W_1 = \rho_1 \cdot \Delta V_1;\ \Delta W_2 = \rho_2 \cdot \Delta V_2, \dots.$$

Let the coördinates of the center of gravity of the piece be x_i, y_i, z_i. Substituting in (15), (16), and (17), we obtain the coördinates $\bar{x}, \bar{y}, \bar{z}$ of the center of gravity.

$$\bar{x} = \frac{\rho_1 x_1 \Delta V_1 + \rho_2 x_2 \Delta V_2 + \dots \rho_n x_n \Delta V_n}{\rho_1 \Delta V_1 + \rho_2 \Delta V_2 + \dots \rho_n \Delta V_n}$$

$$= \frac{\Sigma \rho_i x_i \Delta V_i}{\Sigma \rho_i \Delta V_i} \text{ (where } i = 1, 2, \dots n). \quad . \quad . \quad . \quad (15')$$

Similarly,

$$\bar{y} = \frac{\rho_1 y_1 \Delta V_1 + \rho_2 y_2 \Delta V_2 + \dots \rho_n y_n \Delta V_n}{\rho_1 \Delta V_1 + \rho_2 \Delta V_2 + \dots \rho_n \Delta V_n}.$$

$$\bar{y} = \frac{\Sigma \rho_i y_i \Delta V_i}{\Sigma \rho_i \Delta V_i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16')$$

$$\bar{z} = \frac{\Sigma \rho_i z_i \Delta V_i}{\Sigma \rho_i \Delta V_i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17')$$

Let $\Delta V_i \doteq 0$, now, in such a way that it always includes x_i, y_i, z_i; then since $\lim_{\Delta V_i \doteq 0} (\Delta V_i) = dV_i$, (15′), (16′), (17′) become

$$\bar{x}=\frac{\int x\rho\,dV}{\int\rho\,dV}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

$$\bar{y}=\frac{\int y\rho\,dV}{\int\rho\,dV}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

$$\bar{z}=\frac{\int z\rho\,dV}{\int\rho\,dV}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

the integrations extending over the entire body. We shall now discuss *seriatim* the various quantities entering equations (19), (20), and (21).

96. The Weight per Cubic Unit ρ. — In many problems the mass is homogeneous. In this case, ρ is constant, and may be canceled out of the numerator and denominator. If the density is *variable*, the integrations in general can be performed, provided the law of variation is known, and the density of each point is a function of its coördinates. The law of variation of the density will, in some cases, suggest the method of procedure.

For instance, if the density vary as a function of the distance from a given point, it is advisable to use polar coördinates and to choose this point as a pole. If the density vary as a function of the distance from a line, this line may be taken as one of the coördinate axes and cartesian coördinates employed.

97. The Quantity dV. — Some remarks on the application of the calculus: It is difficult to make suggestions that are general, rigorous, and, at the same time, helpful to the average engineering student for the applications of the calculus to physical problems. A common fallacy entertained by beginners is that calculus deals with small quantities. Now "small" is a relative term. A quantity is small only when compared with another. The truth is that in most applications of the integral calculus to actual physical problems, the definite integral is the limit of a sum of an indefinite number of infinitesimals, and an *infinitesimal is a*

variable, (not necessarily small,) *whose limit is zero.* This is based on the following theorem* proved in works on the calculus:

If $\alpha_1 + \alpha_2 + \cdots \alpha_n$ is a sum of infinitesimals that approaches a finite limit as n increases indefinitely; and if $\beta_1 + \beta_2 + \cdots \beta_n$ is another sum of infinitesimals such that

$$\frac{\beta_1}{\alpha_1} = 1 + i_1\,; \ \frac{\beta_2}{\alpha_2} = 1 + i_2\,; \ \cdots; \ \frac{\beta_n}{\alpha_n} = 1 + i_n\,;$$

where $i_1, i_2, \cdots i_n$ are infinitesimals; then when n increases indefinitely, $\lim(\alpha_1 + \alpha_2 + \cdots \alpha_n) = \lim(\beta_1 + \beta_2 + \cdots \beta_n)$.

Because the method is so important and so widely applied, we shall illustrate its application by a solution of the problem of finding the area included by a plane curve, the x-axis, and two ordinates.

Let the equation of the curve be $y = f(x)$.

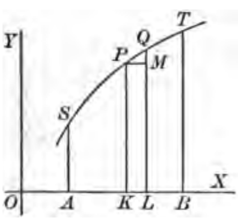


FIG. 56

Let $OA = a$,
$OB = b$.

Divide the area under consideration into n strips by drawing ordinates.

Let the width of each strip $= \Delta x$.
Let $\beta_k = k$th strip $PQKL$.
Let $\alpha_k =$ area $PKLM$.
Let $x_k, y_k =$ coördinates of P.

Now, α_k, β_k, and area PMQ are infinitesimals, for, as n increases indefinitely, they are variables whose limit is zero.

Moreover, $\lim(\alpha_1 + \alpha_2 + \cdots \alpha_n)$ is a finite quantity, viz., the area $ABTS$. Also, $\beta_k - \alpha_k =$ an infinitesimal, $i_k\,\alpha_k$ (say).

$$\therefore \frac{\beta_k}{\alpha_k} = 1 + i_k.$$

Hence all the requirements of the theorem are satisfied.

Hence, $\lim\limits_{n \doteq \infty} (\beta_1 + \beta_2 + \cdots \beta_n)$, the quantity sought,

$$= \lim_{n \doteq \infty} (\alpha_1 + \alpha_2 + \cdots \alpha_n).$$

But $\alpha_1 = y_1 \Delta x$; $\quad \alpha_2 = y_2 \Delta x$; $\quad \cdots \alpha_n = y_n \Delta x$;

i.e. $\alpha_1 = f(x_1)\Delta x$; $\alpha_2 = f(x_2)\Delta x$; $\cdots \alpha_n = f(x_n)\Delta x$.

Moreover, it is proved in the calculus that

*For proofs of this theorem see Williamson's *Differential Calculus*, p. 41, Osgood's *Differential and Integral Calculus*, p. 164.

$\lim (f(x_1)\Delta x + f(x_2)\Delta x + \cdots f(x_n)\Delta x)$ may be found by integrating the indefinite integral $f(x)dx$, substituting the largest and smallest value of x in the result, and subtracting. That is in our case

$$\lim(f(x_1)\Delta x + f(x_2)\Delta x \cdots f(x_n)\Delta x) = \int_a^b f(x)dx.$$

In this problem, which is typical, note that

1. We have chosen as $\alpha_1, \alpha_2, \cdots \alpha_n$ *infinitesimals whose expressions we know.*

2. We have chosen as $\beta_1, \beta_2, \cdots \beta_n$ *infinitesimals whose sum we wish to find*, so related to $\alpha_1, \alpha_2, \cdots \alpha_n$, that $\frac{\beta_k}{\alpha_k} = 1 + i_k$.

3. We have found the sum of the infinitesimals $\alpha_1, \alpha_2, \cdots \alpha_n$, by integrations.

4. We have equated by the theorem of this article the limit of the sum of α's to the limit of the sum of the β's.

The efficiency with which one applies the methods of the calculus depends largely on the choice of the infinitesimals $\alpha_1, \alpha_2, \cdots \alpha_n$, which we shall henceforth call the differential elements. In our problem the differential element is dV.

98. The Quantity dV. — We shall limit ourselves to finding the center of gravity of

(1) A curve of uniform cross section.

(2) A plane area of uniform thickness.

(3) A surface of revolution of uniform thickness.

(4) A solid of revolution of uniform density, the circular sections of which are perpendicular to the axis of revolution.

(5) Any solid.

We shall consider these *seriatim*, quoting without proof such formulæ from the calculus as we need.

(1) *A curve of uniform cross section* a. Choose $dV = ads$, where ds is the differential length of the curve. Since a is constant, it will cancel out of the numerator and denominator. Accordingly, we shall write

$$dV = ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad . \quad (22)$$

(if the equation of the curve is in rectangular coördinates, Fig. 57).

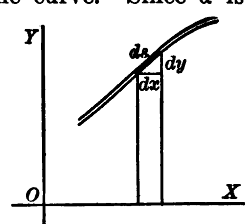


FIG. 57

$$dV = ds = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\,dr \quad . \quad (23)$$

(if the equation of the curve is in polar coördinates, Fig. 58).

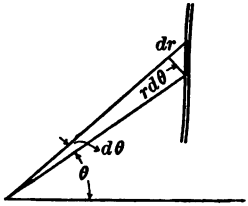


FIG. 58

(2) *An area bounded by plane curves.*

Choose

$$dV = dx\,dy \quad \cdot \quad \cdot \quad (24)$$

(if the equations of the curves are given in rectangular coördinates, Fig. 59)

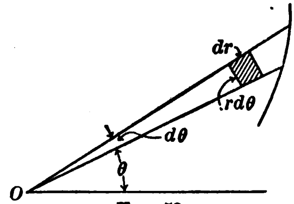


FIG. 59

$$dV = r\,d\theta\,dr \quad \cdot \quad \cdot \quad (25)$$

(if the equations of the curves are given in polar coördinates, Fig. 60).

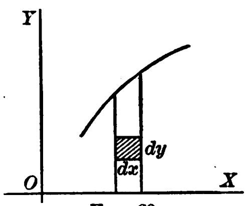


FIG. 60

(3) *A surface of revolution* is the surface generated by revolving a plane curve around a line, or axis, in the plane of the curve. Let the equation of the curve be given in rectangular coördinates. If the axis of revolution coincides with the x-axis, choose

$$dV = 2\,\pi y ds = 2\,\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx,$$

or

$$dV = 2\,\pi y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

(4) *A solid of revolution* is a solid generated by the revolution of a plane area around an axis in its plane. If the plane area is bounded by the x-axis, two ordinates, and a plane curve, choose

$$dV = \pi y^2 dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

(5) *Any solid.*

We may choose as elementary volume the rectangular parallelopiped $dx\,dy\,dz$.

Hence (Fig. 61),

$$dV = dx\,dy\,dz. \quad . \quad . \quad . \quad (28)$$

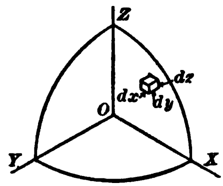


FIG. 61

Or, we may cut the body (Fig. 62) by two

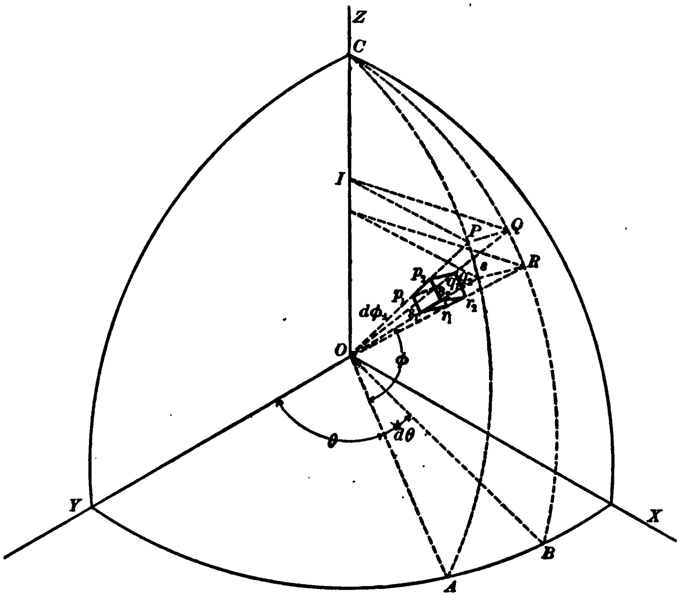


FIG. 62

planes AOC and BOC. On the arc AC, the intersection of the plane and the surface, choose the point P, and pass a plane through it parallel to the xy-plane cutting OC in I and BC in Q. Similarly, through S, on arc AC, pass a plane parallel to the xy-plane cutting CB at R. Draw OP, OQ, OR, and OS; then if $\angle YOA = \theta$, $\angle AOB = d\theta$, and if $\angle AOS = \phi$, then $\angle SOP = d\phi$.

Consider now as our elementary volume, the curvilinear parallelopiped $p_1q_1r_1s_1 - p_2q_2r_2s_2$. Let r, θ, ϕ, be the polar coördinates of p_1, then

$$p_1q_1 = r \cos \phi \, d\theta; \;\; p_1p_2 = dr; \;\; p_1s_1 = r \, d\phi.$$

$$\therefore dV = r \cdot d\phi \cdot r \cos \phi \, d\theta \cdot dr = r^2 \cos \phi \, d\phi \, d\theta \, dr.$$

We are now in position to choose a suitable dV, a suitable ρ, and, having substituted them in equations (19), (20), (21), and performed the necessary integrations, to find $\bar{x}$, $\bar{y}$, $\bar{z}$, the coördinates of the center of gravity of these bodies.

In the articles which follow we shall solve problems illustrating each of the five cases we have treated. Each solution is followed by a list of exercises that illustrate further the methods employed.

99. To find the Center of Gravity of a Circular Arc. — Let AB (Fig. 63) be a circular arc described about O as center with radius a, subtending an angle $AOB = 2\alpha$. At P, any point on the arc, choose an elementary volume,

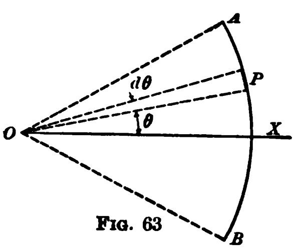


Fig. 63

$$dV = ds = a\,d\theta.$$

Let O be the origin and OX, a line bisecting the arc, be the x-axis. Substituting in (19)

$$\bar{x} = \frac{\int x\,ds}{\int ds} = \frac{\int_0^\alpha a\cos\theta \cdot a\,d\theta}{\int_0^\alpha a\,d\theta} = \frac{a\sin\theta\Big]_0^\alpha}{\theta\Big]_0^\alpha} = \frac{a\sin\alpha}{\alpha}. \quad . \quad . \quad (29)$$

$\bar{y} = 0.$

100. Exercises

1. Find the center of gravity of a circular arc which subtends an angle of 60° at the center of the circle.

Ans. On a diameter bisecting the arc and at a distance $\dfrac{3a}{\pi}$ from the center.

2. Find the center of gravity of the quadrantal arc of a circle.

Ans. On a diameter bisecting the arc and at a distance $\frac{2\sqrt{2}\,a}{\pi}$ from the center.

3. Find the center of gravity of a semicircular arc.

4. Find the center of gravity of a straight rod of uniform thickness, the density of which varies as the nth power of the distance of each point from one end. Let a = length of rod.

Ans. $\bar{x} = \frac{n+1}{n+2}a$ from the end of least density.

5. Find the center of gravity of a quadrantal arc of a circle, OX and OY being the bounding radii.

101. Find the Center of Gravity of the Area bounded by the x-axis and a Quadrant of the Ellipse whose Equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let us choose $dV = dx\,dy$.

$$\therefore \bar{x} = \frac{\int\int x\,dx\,dy}{\int\int dx\,dy}.$$

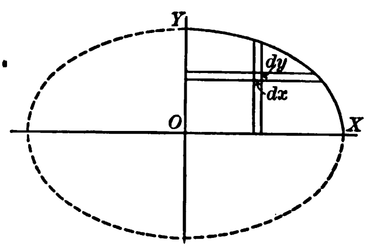


FIG. 64

Now

$\int\int dx\,dy = \pi\frac{ab}{4} = \frac{1}{4}$ of the area of an ellipse.

Let us integrate $\int\int x\,dx\,dy$, first with regard to x, keeping y constant; the limits of this integration evidently are

$$\frac{a}{b}\sqrt{(b^2-y^2)} \text{ and } 0.$$

$$\therefore \bar{x} = \frac{\frac{1}{2}\int_0^b \frac{a^2}{b^2}(b^2-y^2)dy}{\pi\frac{ab}{4}} = \frac{4}{3}\frac{a}{\pi}.$$

The student may find $\bar{y}$.

102. To find the center of gravity of a circular area of uniform thickness the density of which varies as the distance from a point on the circumference.

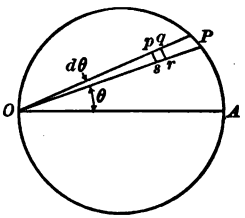


FIG. 65

Let the diameter of the circumference $= 2a$.

Let O be the pole.

Let OA be the initial line.

If r, θ be the coördinates of any elementary area $pqrs$, then

$$dV = r\,d\theta\,dr, \text{ and } \rho = kr.$$

Equation (19) becomes

$$\bar{x} = \frac{\int\int r^3 \cos\theta\,dr\,d\theta}{\int\int r^2\,dr\,d\theta}.$$

Integrating first with respect to r, it is evident that the limits, when θ is constant, are $OP = 2a\cos\theta$ and 0.

$$\therefore \bar{x} = \frac{\int_0^{\frac{\pi}{2}} \frac{(2a)^4 \cos^5\theta\,d\theta}{4}}{\int_0^{\frac{\pi}{2}} \frac{(2a)^3}{3}\cos^3\theta\,d\theta} = \frac{6}{5}a.$$

$$\bar{y} = 0.$$

103. To find the center of gravity of the area of uniform density included between a parabola and a straight line which passes through the vertex.

Let the equations of the curves be $y^2 = 4ax$

and $y = mx$.

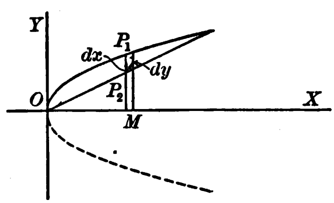


FIG. 66

Then

$$\bar{x} = \frac{\int\int x\,dy\,dx}{\int\int dy\,dx}, \qquad \bar{y} = \frac{\int\int y\,dy\,dx}{\int\int dy\,dx}.$$

We may integrate first with respect to y, the limits of this integration being $MP_1 = \sqrt{4\,ax}$ and $MP_2 = mx$.

Let $x_1 = \dfrac{4\,a}{m^2}$ be the x-coördinate of the intersection of the two curves.

$$\therefore\ \bar{x} = \frac{\int_0^{x_1} x(\sqrt{4\,ax} - mx)dx}{\int_0^{x_1} (\sqrt{4\,ax} - mx)dx},$$

$$y = \frac{1}{2}\,\frac{\int_0^{x_1} (4\,ax - m^2x^2)\,dx}{\int_0^{x_1} (\sqrt{4\,ax} - mx)dx}.$$

$$\therefore\ \bar{x} = \frac{8\,a}{5\,m^2}.$$

The student may solve for $\bar{y}$.

104. **Exercises**

1. Find the center of gravity of the area included between the x-axis, $y^2 = 8\,x$, and the ordinate whose length is 4.

Ans. $\bar{x} = \frac{6}{5}$, $\bar{y} =$.

2. Find the center of gravity of the area included between the parabola $y^2 = 8\,x$ and the straight line $y = x$ if the density of any point varies as its distance from the x-axis.

Ans. $\bar{x} = 4$, $\bar{y} =$.

3. Find the center of gravity of a circular sector of uniform density the arc of which subtends 60°.

Ans. $\dfrac{2\,a}{\pi}$ from the center.

105. Find the center of gravity of a solid of revolution generated by revolving the parabola $y^2 = 4\,ax$ around the x-axis. Substituting from equation (27),

$$\bar{x} = \frac{\int_0^h xy^2\,dx}{\int_0^h y^2\,dx} = \frac{\int_0^h x^2\,dx}{\int_0^h x\,dx} = \frac{2}{3}\,h,$$

where h is the altitude of the paraboloid of revolution.

106. **Exercises**

1. A hollow cylinder with one end closed is made of cast iron one inch thick. Find its center of gravity, assuming the cylinder is 3 feet high and has an inside radius of 9 inches.

Ans. 19.8 inches from open end, on axis.

2. From a sphere of radius R is removed a sphere of radius r, the distance between their centers being h. Find the center of gravity of the remainder.

Ans. It is on the line joining their centers at a distance $\dfrac{hr^3}{R^3 - r^3}$ from the center of the larger sphere.

3. Find the center of gravity of a sector of a circle of radius r, the arc of which subtends an angle of 2α.

Ans. If the axis of x is the diameter of the circle that bisects the sector and the center is the origin,

$$\bar{x} = \tfrac{2}{3} r \frac{\sin \alpha}{\alpha}.$$

4. Find the center of gravity of the area inclosed between the parabola $y^2 = 4ax$ and the double ordinate corresponding to the abscissa $x = a$.

Ans. $\bar{x} = \dfrac{3a}{5}$, $\bar{y} = 0$.

5. Find the center of gravity of the area between the curves $y^2 = 8x$ and $x^2 = 8y$.

Ans. $\bar{x} = \bar{y} = \frac{18}{5}$.

6. Find the center of gravity of the area formed by removing from a circle a sector the arc of which subtends 90°.

Hint. — Apply Art. 93.

7. Find the center of gravity of a circular area whose density varies as the square of the distance from a point on the circumference.

8. Find the distance from the vertex to the center of gravity of a right circular homogeneous cone.

Ans. $\frac{3}{4}$ the altitude of the cone.

9. Find the center of gravity of a frustum of a homogeneous right cone, the radii of the bases being 4 and 2, and its height 3 inches.

10. Prove: The center of gravity of a homogeneous triangular pyramid is on a line joining the center of gravity of the base and the vertex of the pyramid, and $\frac{3}{4}$ the distance from the vertex to the center of gravity of the base.

11. Find the center of gravity of a right circular cone when the density of the cone varies as the distance from a plane through the vertex and parallel to the base.

Ans. $\bar{x} = \frac{4}{5}$ altitude.

12. Find the center of gravity of a hemispherical solid.

Hint. — Revolve the quadrant of a circle around one of the radii bounding it.

13. A half ellipsoid is formed by the revolution of the quadrant of an ellipse about its major axis; find the distance of the center of gravity of the solid from the center of the ellipse.

Ans. $\bar{x} = \frac{3a}{8}$, when $a =$ semi-major axis.

14. Find the center of gravity of a hemispherical shell.

15. Find, using the result of exercise 14, the center of gravity of a hemispherical solid if the density varies as the square of the distance from the center.

Hint. — Use as elementary volume a hemispherical shell, and consider the hemisphere as made up of a series of concentric hemispherical shells.

Ans. $\bar{x} = \frac{5a}{12}$.

16. Find the center of gravity of that part of a sphere contained between two parallel planes whose distances from the center of the sphere equal $\frac{1}{4}a$ and $\frac{3}{4}a$, respectively, where $a =$ radius of the sphere.

107. The Theorems of Pappus or Guldinus.

FIRST THEOREM. — *A plane area, bounded by a closed curve, revolving about an axis in its plane but outside the area, generates a solid, the volume of which equals the product of the revolving area and the distance traveled by its center of gravity.*

SECOND THEOREM. — *An arc of a plane curve revolving about an axis lying in the plane of the curve, but not intersecting it generates a surface of revolution, the area of which equals the product of the length of the revolving arc and the length of the path described by its center of gravity.*

In the proofs which follow we assume the x-axis as the axis of revolution. In the proof of the *first* theorem, we assume that any line parallel to the y-axis cuts the curve bounding the area in two points only.

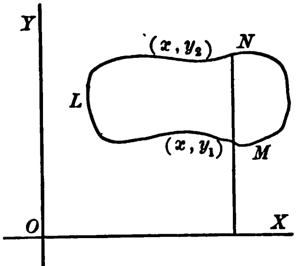


FIG. 67

To prove the first theorem : Let the plane area, bounded by the curve LMN (Fig. 67) revolve about the x-axis, generating a volume V Let the area inclosed by $LMN = A$

Let the ordinate of the center of gravity of $A = \bar{y}$ (when A lies in the xy-plane). Let LMN (lying in xy-plane) be cut by any ordinate in the points x, y_1, x, y_2. Then (Art. 95),

$$A \cdot \bar{y} = \int\int_{y_1}^{y_2} y\, dy\, dx = \tfrac{1}{2}\int (y_2^2 - y_1^2) dx. \quad . \quad . \quad . \qquad (a)$$

$$2\pi \bar{y} \cdot A = \pi \int (y_2^2 - y_1^2) dx = V. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

Q.E.D.

To prove the second theorem :

Let the plane curve AB revolve around the x-axis, generating the surface of revolution of area S.

Let ds = elementary arc of the revolving curve,

$\bar{y}$ = ordinate of center of gravity of the curve AB,

s = length of the curve AB.

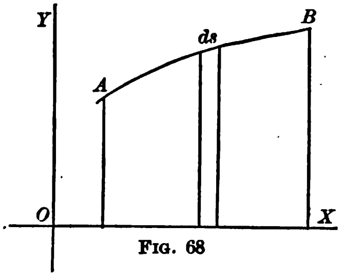


FIG. 68

Then (Art. 95),

$$s\bar{y} = \int y\, ds. \quad \text{. . . . .} \quad (b)$$

But $$2\pi \int y\, ds = S.$$

$$\therefore 2\pi\bar{y}s = S. \quad \text{. . . . .} \quad (31)$$

Q.E.D.

COROLLARY. — *It is evident that instead of multiplying equations* (*a*) *and* (*b*) *of this article by* 2π, *we may multiply them by* θ, *any angle* (*expressed in radians*). *The theorems then give respectively the volume and surface generated in revolving the arc or the area through the angle* θ.

108. Exercises

1. Find the surface and volume of a circular ring with circular cross section; the internal diameter of the ring is 12 inches; the diameter of the cross section is 3 inches.

Ans. $S = 444.1$ sq. in.; $V = 333.1$ cu. in.

NOTE. — Such a ring is called an anchor ring.

2. An equilateral triangle revolves around its base, whose length is a. Find the surface and the volume of the solid thus generated.

Ans. $S = \pi a^2\sqrt{3}$; $V = \dfrac{\pi a^3}{4}$.

3. An equilateral triangle, each side of which $= a$, revolves around a line parallel to one side of the triangle and at a distance h from it. Find the surface, and the volume thus generated,

(1) if the line is on the same side of the base as the vertex and $h = \frac{3}{2} a$; (2) if the line is on the side of the base opposite the vertex and $h = \frac{3}{2} a$.

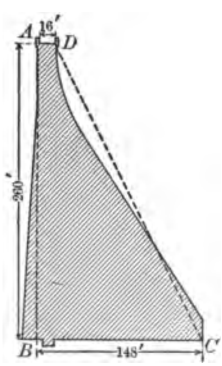
4. Let AB be the quadrant of a circle referred to the bounding radii as axes. Assuming that we know the area of a hemisphere, calculate the center of gravity of the quadrant.

Ans. $\bar{x} = \bar{y} = \dfrac{2a}{\pi}$.

5. Assuming that we know the volume of a sphere, find, using Art. 107, the center of gravity of a semicircular area.

6. Using the result of Art. 101, find the volume of a semi-ellipsoid formed by revolving the quadrant of an ellipse around its minor axis.

7. The Roosevelt masonry dam, Salt River, Arizona, is built in the form of a circular arc 644.03 feet long, measured along the middle of the top of the dam. This arc subtends an angle of 90° at the center of the circle. The maximum cross section of the dam is shown in the figure. If we assume that the dam is of uniform cross section along a radial line, and that the area of the cross section is equivalent to the trapezoid $ABCD$ in the figure, the angles at A and at B being right angles, and that the center of gravity of the section and trapezoid coincide, find the volume of the dam, the vertical side of the dam facing the center of the circle.



NOTE. — A complete description of this dam may be found in the *Engineering News*, Jan., 1905.

Hint. — Show that the center of gravity of the trapezoid is 49.85 feet from AB. Apply the method of Art. 107, find the volume = 560,455 cu. yds.

CHAPTER VII

FRICTION

109. In considering the reaction of bodies on each other (Art. 42) we defined a smooth surface as one that offers normal resistance only. But actual surfaces are not smooth; they have certain depressions and projections, and if two surfaces are in contact with each other, the projections of one of them enter into the depressions of the other, thereby offering resistance to motion in a tangential direction. This resistance is called the force of friction.

The direction of friction between a body and the surface over which it moves or tends to move is along the surface at the point of contact and opposite to that direction in which the tendency to motion is. For example, if a body of weight W, lying on a horizontal plane, is acted upon by a force P, parallel to the plane, the force of friction acts in the direction opposite to P.

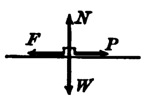


FIG. 69

110. The magnitude of the force of friction depends on the nature of the bodies in contact and the force applied to them. Friction will prevent a body from moving if it can, but only so much will be brought into play as is necessary to prevent motion.

We shall at present consider only *static* friction, that is, the friction of bodies at rest, and we shall be interested only in those problems in which the equilibrium of bodies is about to be broken; that is, when the force of friction between any two given bodies is the greatest possible. This is called *limiting friction.*

For example, if a weight W is lying on a plane, as in the

last article, and P is smaller than a certain force, friction will maintain equilibrium. If P be increased until the body is just on the point of moving, the friction will be the greatest possible. This is limiting friction. If P is still further increased, friction cannot prevent motion of the body.

111. Laws of Friction. — It has been found by experiment that:

I. For any two given surfaces (under the same condition) the ratio of the *normal pressure* to the *limiting friction* is *constant.*

II. If the surfaces are not deformed, the magnitude of limiting friction is *independent* of the *extent* of *surface* in *contact.*

III. The direction of the friction is tangent to the surface upon which the body tends to move and opposite to the direction of tendency to motion.

Discussion of these Laws. — Law I gives the magnitude of the force. Law III its direction and sense. Law I is usually written thus:

Let $N =$ normal pressure between two bodies.

Let $F =$ force of friction.

Then, $\frac{F}{N} = f$ (a constant).

$$\therefore F = fN. \qquad (32)$$

This equation is true only when slipping is about to take place.

112. The quantity f is called the *coefficient of friction*, and, as has been said, is constant for two given surfaces. It is determined by experiment and is in general a fraction. For wood on wood it varies from 0.25 for some woods to 0.50 for others. For wood on wood soaped, f varies from 0.04 for some woods to 0.20 for others. For the best lubricated and smoothest surfaces, f is about 0.03.

113. Table of Friction of Plane Surfaces which Have Been Some Time in Contact

(*Morin*)

Kind of Surfaces in Contact	*Disposition of the Fibers*	*Condition of the Surfaces*	*f*	ϕ ° ′
Oak on oak	Parallel	Without unguent	0.62	31–48
	Parallel	Rubbed with dry soap	0.44	23–45
	Perpendicular	Without unguent	0.54	28–22
	Perpendicular	Moistened with water	0.71	35–22
	Wood upright on wood flatwise	Without unguent	0.43	23–16
Tanned leather on oak	The leather flatwise	Without unguent	0.61	31–23
	The leather on edge	Without unguent	0.43	23–16
	. . . .	Moistened with water	0.79	38–19
Black curried leather or belt — On plane oak surface	Parallel	Without unguent	0.74	36–30
Black curried leather or belt — On oak drum	Perpendicular	Without unguent	0.47	25–10
Hemp cord on oak	Parallel	Without unguent	0.80	38–40
Iron on oak	Parallel	Without unguent	0.62	31–48
	Parallel	Moistened with water	0.65	33–01
Cast iron on oak	Parallel	Moistened with water	0.65	33–01
Brass on oak	Parallel	Without unguent	0.62	31–48
Black curried leather, or belt upon cast iron pulley	Flatwise	Without unguent	0.28	15–39
	Flatwise	Moistened with water	0.38	20–48
Cast iron upon cast iron	Flatwise	Without unguent	0.16*	9–05

* The surfaces being somewhat unctuous (oily or greasy).

114. The Angle of Friction. — Imagine a body lying on a plane, the inclination of which can be changed, such as the lid of a desk. So long as a plane is horizontal, the body will not move; if, however, the inclination is increased, a position will be reached eventually such that the body will begin to slide down the plane. The *inclination* of the plane when the body is *just about to slip* is called the *angle of friction*, or *angle of repose.*

115. To find the Angle of Friction. — Let a small body, whose weight is W, rest on a rough plane inclined to the horizon by an angle ϕ, such that the body is about to move down the plane. Then ϕ = angle of friction. The body is in equilibrium under three forces, N, the normal pressure, $F = fN$, the friction, and $W =$ the weight of the body.

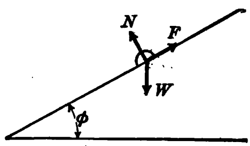


FIG. 70

Resolving horizontally,

$$F \cos \phi = N \sin \phi.$$

But, $$F = fN.$$

Hence, $$f = \tan \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

116. The Total Reaction. — A rough plane reacts on a body resting on it, under the action of forces, in two ways: there is a normal reaction and friction. These forces may be replaced by a single force R, which we call the *Total Reaction* of the plane. If R makes an angle α with the normal, we have

$$R \sin \alpha = F;$$

$$R \cos \alpha = N;$$

$$\tan \alpha = \frac{F}{N},$$

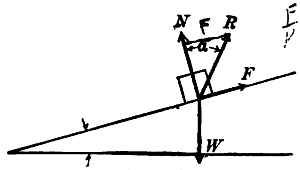


FIG. 71

whatever F may be. But in *limiting equilibrium* $\frac{F}{N} = f$ (Art. 111). Therefore, *in this case*, $\tan \alpha = \tan \phi$, and hence $\alpha = \phi$, where ϕ is the angle of friction.

If the total reaction of the plane makes an angle with the normal less than or equal to the angle of friction, the actual friction is less than or equal to the limiting friction, and the point of contact will not slip. But if, in order to maintain the equilibrium of the point of contact, a total reaction is necessary whose direction makes an angle with the normal greater than the angle of friction, slipping will ensue.

In general, we shall be interested in the magnitude and direction of the total reaction only in the case of limiting friction.

Then $$\alpha = \phi,$$

and $$R^2(\sin^2\phi + \cos^2\phi) = F^2 + N^2 = N^2(1 + f^2).$$

$$\therefore R = N\sqrt{1 + f^2}.$$

117. Remark.—In what follows f, N, R, and ϕ will designate respectively the coefficient of friction, the normal reaction of the plane, the total reaction of the plane, and the angle of friction. It is evident that if the body lie on any surface, the plane tangent to the surface may be considered as an inclined plane. Also, that so long as the system of forces is such as to allow R to take a direction with the normal equal to the angle of friction, the body will be in limiting equilibrium. In the preceding theory, we have assumed that the equilibrium is broken by sliding. In some instances equilibrium will be broken by rolling, or by rolling and sliding. We shall take these cases up later. In the problems which immediately follow, we shall assume that equilibrium is broken by sliding only.

118. To illustrate the preceding theory, consider the problems: A body, whose weight is W, resting on a plane inclined 60° to the horizon, is on the point of moving down the plane. It is prevented from slipping by friction and a horizontal force equal to the weight of the body. Find the coefficient of friction. Resolve the forces acting on the body along the plane and perpendicular to the plane.

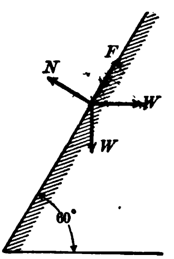


Fig. 72

$$F = W(\sin 60° - \cos 60°).$$

$$F = fN = fW(\cos 60° + \sin 60°).$$

$$\therefore f = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}.$$

119. A uniform bar AA' rests with one end A on a rough horizontal plane and the other end against a rough vertical wall. Let the coefficients of friction of the bar and the plane and the bar and the wall be respectively f and f'. Find the angle θ (see Fig. 73) when the point A is about to slip.

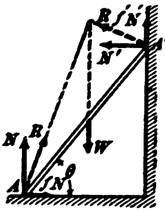


FIG. 73

We shall assume that A and A' are in limiting equilibrium (see next article).

Taking moments about A', and resolving forces vertically and horizontally, if $2L =$ length of the rod, we get,

$$2\,NL\cos\theta - WL\cos\theta = 2\,L\,fN\sin\theta.$$

$$N + f'N' = W.$$

$$fN = N'.$$

$$\therefore \tan\theta = \frac{1-ff'}{2f}.$$

Since $\tan\theta$ is positive, equilibrium is possible if $ff' < 1$.

Or, we may solve the problem graphically. The bar is in equilibrium under three forces, R, R', and W, which, therefore, meet in the point C. Therefore (footnote, exercise 9, Art. 88),

$$2\tan\theta = \cot\phi - \tan\phi = \frac{1}{f} - f' = \frac{1-ff'}{f}.$$

120. The bar in the preceding article will rest in many positions. For, draw the reaction R' intersecting the line of action of the weight at C. Then if the reaction R can pass through C making an angle with N less than ϕ, the bar will be in equilibrium, but not limiting equilibrium. The reader interested may consult Minchin's *Statics*, Vol. I, p. 260, or Routh's *Analytic Statics*, Vol. I, p. 113.

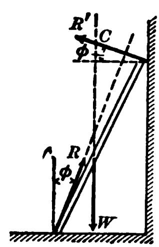


FIG. 74

121. Exercises

1. A horizontal force of 1 pound acts on a body weighing 100 pounds, which rests on a rough horizontal plane. If the coefficient of friction between the body and the plane equals 0.1, find the friction between the bodies. *Ans.* 1 pound.

2. If in exercise 1, the horizontal force be replaced by another of 5 pounds making an angle of 30° with the plane, find the friction.

3. A horizontal force of 10 pounds acts on a body weighing 100 pounds, resting on a rough horizontal plane; if the body is on the point of slipping, find the coefficient of friction.

4. A weight of 500 pounds rests on a plane inclined 30° to the horizon. It is acted upon by a force P inclined to the plane 30°. The coefficient of friction is 0.1. Find P if the body is on the point of sliding down the plane. Find P if the body is on the point of moving up the plane.
Ans. 253.3 pounds when the body is about to move down.

5. A body of weight W rests on a plane inclined at an angle α to the horizon. A force P, making an angle θ with the plane, acts on the body. The coefficient of friction $= f$.

(1) Find the force P if the body is on the point of moving down the plane.

(2) Find the force P in case the body is on the point of moving up the plane. *Ans.* (1) $P = W\dfrac{\sin\alpha - f\cos\alpha}{\cos\theta - f\sin\theta}$,

$$(2)\ P = W\frac{\sin\alpha + f\cos\alpha}{\cos\theta + f\sin\theta}.$$

If $f = \tan\phi$, these answers may be put in the form:

$$(1)\ P = W\frac{\sin(\alpha - \phi)}{\cos(\theta + \phi)},$$

$$(2)\ P = W\frac{\sin(\alpha + \phi)}{\cos(\theta - \phi)}.$$

6. In the preceding problem, find the least possible force P and the corresponding angle θ: (1) which will just prevent the body from slipping down the plane; (2) that will cause the body to be just on the point of slipping up the plane.

Ans. (1) $\theta = -\phi$; $P = W \sin(\alpha - \phi)$.

Hint. — Find derivative of P with regard to θ in the answer to exercise 5, and equate it to zero.

7. Find the direction and the magnitude of the total reaction of the plane in exercise 4.

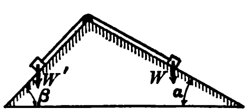
8. Find the least possible force and the angle which its line of action makes with the plane in order that the weight in exercise 4 should be on the point of moving up the plane.

9. A body whose weight is 20 pounds is just sustained on a rough inclined plane by a horizontal force of 2 pounds and a force of 10 pounds along the plane. The coefficient of friction is $\frac{2}{5}$. Find the inclination of the plane.

Ans. If the inclination of the plane is α, $\tan \dfrac{\alpha}{2} = \dfrac{25}{48}$.

Find the direction and the magnitude of total reaction of the plane.

10. W and W' lie on a double inclined plane as in the figure; the inclinations of the two planes are α and β, respectively. The weights are connected by a string passing over a smooth pulley. The coefficient between W and the plane on which W lies is f; the plane on which W' lies is smooth. (a) Find the tension of the string. (b) Find W' in terms of W, when W is about to move up the plane; (c) when W is about to move down the plane.



Ans. (a) $T = W' \sin\beta$.

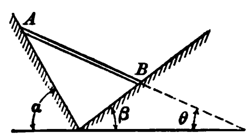
(b) $W' = \dfrac{W(\sin\alpha + f\cos\alpha)}{\sin\beta}$.

11. A heavy rod whose center of gravity divides the rod in the ratio of a to b rests with one end on a horizontal plane.

The other end rests on a smooth cylinder, to which it is tangent. The coefficient of friction between the plane and the rod equals f. The weight of the rod equals W. Find the position in which the rod will rest.

Ans. $(a+b)\tan\theta = b\cot\phi - a\cot\theta$. (See footnote, exercise 9, Art. 88.)

12. A heavy uniform rod AB, of weight W, rests on two rough inclined planes. Find the inclination of the rod θ to the horizon, when the end of the rod B is about to move up the plane, the coefficient of friction between the rod and the planes being f.



Suggestion. — Put at A and at B the total reaction of the planes; then AB is in equilibrium under *the action of three forces.* (See footnote, exercise 9, Art. 88.)

Ans. If $f = \tan\phi$, $2\tan\theta = \cot(\beta+\phi) - \cot(\alpha-\phi)$.

Find the total reaction at A.

NOTE. — Apply Triangle of Forces.

122. The Axiom of Friction. — The equilibrium of a body resting on a rough surface may, when additional force is applied, be broken in one of two ways: viz., leaving out of consideration the case when the body leaves the surface, it may roll about a point or edge of contact, or it may slip. The criterion as to the nature of the motion is found in the *Axiom of Friction:* which is, *that friction will preserve equilibrium if possible* (Art. 110). A more complete statement is that friction will prevent a point of contact from moving if the total reaction at that point can take such direction and magnitude as to preserve its equilibrium. That is, when equilibrium is broken, it will be broken by rolling, rather than by sliding, because friction prevents the motion of the point or the edge of contact from slipping. This comes to saying that the body will roll if the total reaction at the point or edge of contact can make

an angle with the normal less than or equal to the angle of friction (Art. 116).

As an example, consider a cubical block resting on a horizontal plane, acted on by a force P making an angle θ with the horizon, P being applied at the middle point of one of the upper edges of the cube.

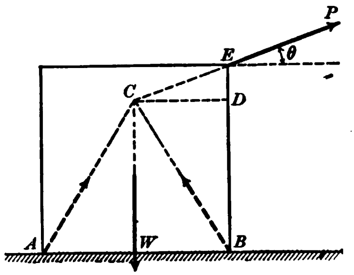


FIG. 75

Let the coefficient of friction between the plane and the cube be $f = \tan\phi$.

Equilibrium may be broken by

(1) rolling around A,
(2) rolling around B,
(3) slipping.

Let us suppose first that the direction of P cuts the line of action of W above AB, and consider *seriatim* (1), (2), and (3).

(1) The cube is in equilibrium under the action of three forces, P, W, and the reaction at A, which must pass through C, the intersection of P and W. Since the normal reaction on the cube is upward, the reaction must take the direction indicated by the arrow. But this is impossible, for in this case the sum of the resolved forces along AB cannot equal zero. Therefore, the cube cannot roll about A.

(2) If it roll about B, the total reaction at B must be able to take a direction so that it passes through the intersection of P and W. Mathematically we can express it thus:

Let CD be perpendicular to BE, the edge of the cube. Then the cube will roll about B if

$$\tan\phi \geqq \tan DBC.$$

But $$\tan DBC = \frac{CD}{BD} = \frac{CD}{2\,CD - CD\tan\theta} = \frac{1}{2 - \tan\theta}.$$

Therefore, it will roll if

$$f = \tan\phi \geqq \frac{1}{2 - \tan\theta};$$

otherwise, it will slip.

$$\therefore 2f - f \tan\theta > 1,$$

$$\frac{2f-1}{f} > \tan\theta.$$

That is, if $\tan\theta < \dfrac{2f-1}{f}$, the cube will roll.

If this condition be satisfied, we can take moments about B and determine P so as to roll the cube around B. Otherwise, the cube will slip.

123. Let us consider one additional problem: A rectangular prism rests on a rough inclined plane with one edge AC of the prism parallel to the line of intersection of the inclined with a horizontal plane. To determine how the equilibrium will be broken when the inclination of the plane is sufficiently increased.

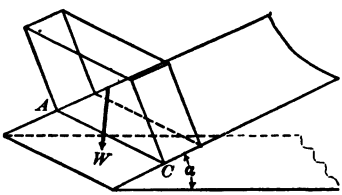


FIG. 76

Let α = inclination of the plane.

ϕ = angle of friction.

$2h$ = altitude of the prism.

$2b$ = length of one side of the base perpendicular to AC.

When the plane is inclined so that the line of action of W intersects the side AC, $\tan\alpha = \dfrac{b}{h}$.

Now the prism cannot roll as long as the line of action of its weight cuts the base.

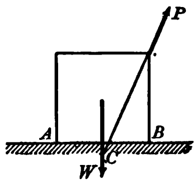
Therefore the body cannot roll as long as $\tan\alpha < \dfrac{b}{h}$.

Under these conditions it will slide if, and only if,

$$\tan\phi < \frac{b}{h}.$$

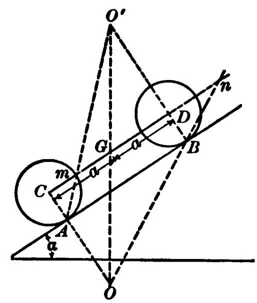
If the $\tan\phi > \dfrac{b}{h}$, the body will roll.

124. **Exercises**



1. Suppose the line of action of P (Art. 122) cut the line of action of W *below* AB. The student may show that the cube will not roll about B, and find the condition that it will roll about A. Also the condition that it will slip. In what direction will it slip?

2. Two equal carriage wheels whose centers are connected by a smooth bar are placed on a rough inclined plane; determine whether the equilibrium of the system will be best preserved by locking the upper or the lower wheel.



Let C and D be the centers of the wheels, and first let the upper wheel be locked. Since there is no friction between the bar CD and the axle at C, the action of the bar on the lower wheel consists of a force through C. The weight of this wheel also acts through C, and therefore the total resistance at A, which is the third force keeping the wheel in equilibrium, must also act through C.

Let G be the center of gravity of the two wheels, and consider the equilibrium of the system formed by them. There are three forces acting on the system, viz., its weight through G, the total resistance at A (which has been proved to act in a line AC), and the total resistance at B. If, then, O is the point of intersection of the vertical through G and of CA, the total resistance at B must act in the line OB.

We shall now determine the inclination at which equilibrium is broken.

Since the upper wheel slips, the angle $DBn = \phi$; also let r equal the radius of each wheel, $CD = 2a$, α equal the inclination of the plane, and $f =$ coefficient of friction. Then,

$$\frac{\tan COG}{\tan COn} = \frac{CG}{Cn}, \text{ or } \frac{\tan \alpha}{f} = \frac{a}{2a + fr},$$

since $Dn = r \tan Dbn = fr$. The inclination of the plane when equilibrium is broken is therefore given by the equation

$$\tan \alpha = \frac{fa}{2a + fr}.$$

Again, suppose the lower wheel to be locked. In this case the total resistance at B acts in the line BD, and that at A acts in the line AO', O' being the intersection of BD with OG. If α' is the new inclination at which equilibrium is broken, we have, since the angle CAO' equals ϕ,

$$\frac{\tan \alpha'}{f} = \frac{DG}{Dm} = \frac{a}{2a - fr};$$

or

$$\tan \alpha' = \frac{fa}{2a - fr}.$$

Now it is clear that α' is greater than α, and that, consequently, equilibrium will be safer when the lower wheel is locked than when the upper wheel is locked; that is, it would be safer to put a brake on the front wheel than on the rear wheel.

3. A pyramid with a square base rests on a rough inclined plane with one side parallel to the intersection of the inclined with the horizontal plane. The altitude of the pyramid $= 4$ feet, and the length of one side of the base $= 18$ inches; $f = 0.2$. If the inclination of the plane is increased until equilibrium is broken, find the nature of the beginning motion and the inclination of the plane at that time.

CHAPTER VIII

FLEXIBLE CORDS

125. The Pulley. — A pulley consists of a circular wheel which may turn about an axle through its center. By passing a flexible cord around the pulley it is possible to change the direction of the force along the cord.

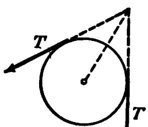


FIG. 77

We shall assume that there is no friction between the wheel and its axle, and that the cord does not slip on the pulley. When the system is in equilibrium, the tension on one side of the pulley equals the tension of the cord on the other side, as will appear by taking moments about the center of the axle. Moreover, the resultant of the tensions bisects the angle between

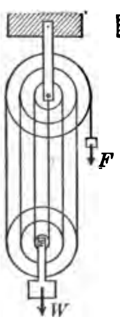


FIG. 78 *a*

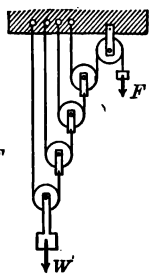


FIG. 78 *b*

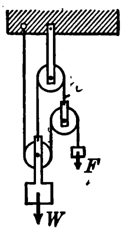


FIG. 78 c

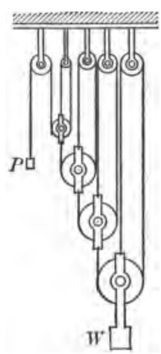


FIG. 78 *d*

them. Pulleys may be combined and arranged in a variety of ways, some of which are shown in Figs. 78 *a*, 78 *b*, 78 *c*, and 78 *d*.

We shall neglect the weights of the cord and the pulleys, and shall assume that the strands of the cord not in contact with the pulley are parallel, and that the system is in equilibrium. Suppose now a force F supports a weight W, F and W applied as in the figures. To find the relation between F and W, consider the equilibrium of those pulleys that can move parallel to the strands of the cord, and equate to zero the forces acting parallel to this direction.

For example, consider Fig. 78 *a*, in which a continuous cord passes successively around the pulleys. The tension of the cord $= F$.

$$\therefore \; W = 6\,F.$$

If there are n parallel strands

$$W = nF.$$

126. Exercise. — The student may find the relation between F and W, if the arrangement of the pulleys is as shown in Figs. 78 *b*, 78 *c*, and 78 *d*.

127. The Suspension Bridge. — In a suspension bridge, a floor structure supposed horizontal and of uniform weight per unit length is suspended by means of vertical rods from two cables assumed weightless, which are fastened to two abutments A and B (Fig. 79). Our problem is to find the form which the cable will assume and the tension in any part of the cable. Let the distance between any two vertical rods be equal. Then each rod sustains the same weight w. Furthermore, suppose the distance between the rods becomes so small that the polygon which the cable assumes is a smooth curve. To find its equation, consider that part of the cable extending from the lowest point O to any point P on the cable. The forces acting on this rigid body are: H, the tension at O,

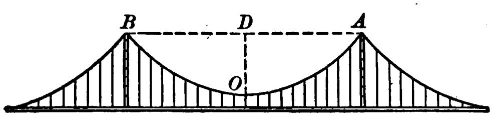


FIG. 79

acting horizontally; T, the tension at the point P, making an angle ϕ with the horizontal; and the weight supported by the cable, equaling the weight per foot times the horizontal projection x.

Then, resolving horizontally, $(\Sigma X_i = 0)$,

$$T \cos \phi = H. \quad \cdots\cdots \quad (a)$$

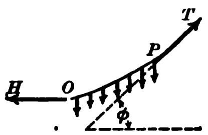


FIG. 80 *a*

Resolving vertically, $(\Sigma Y_i = 0)$,

$$T \sin \phi = wx. \quad \cdots\cdots \quad (b)$$

Dividing (b) by (a) we get,

$$\tan \phi = \frac{wx}{H}. \quad \cdots\cdots \quad (c)$$

But

$$\tan \phi = \frac{dy}{dx}.$$

$$\therefore \frac{dy}{dx} = \frac{wx}{H}.$$

Integrating, we get

$$y = \frac{wx^2}{2H} + C.$$

Since $x = 0$, when $y = 0$, C will equal 0, and we have

$$x^2 = \frac{2H}{w} y,$$

which is the equation of a parabola with its axis vertical and its vertex at the origin.

Let us next find the tension H. Consider as a rigid body that part of the cable extending from O to A, the coördinates of which we shall assume to be a, h. Then the forces acting on this rigid body are wa, the total weight; H, the horizontal tension; T_A, the tension at A. Since the body is in equilibrium, these forces must intersect at a point whose coördinates are $\frac{a}{2}$, 0. Let α equal the angle that T_A makes with the x-axis. Then,

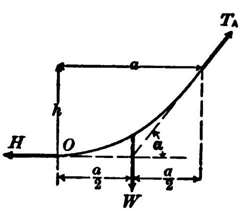


FIG. 80 *b*

$$\tan\alpha = \frac{2h}{a} \text{ (geometry).}$$

Now $$\tan\phi = \frac{wx}{H} \text{ (equation } (c)),$$

and when $$x = a,\ \phi = \alpha.$$

$$\therefore \frac{wa}{H} = \frac{2h}{a}$$

or $$H = \frac{wa^2}{2h}.$$

Resolving horizontally,

$$T_A \cos\alpha = H.$$

$$\therefore T_A = \frac{H}{\cos\alpha} = \frac{wa}{2h}\sqrt{4h^2 + a^2}.$$

Squaring equations (a) and (b), and adding, the expression for the tension T at any point becomes

$$T = \sqrt{H^2 + (wx)^2}$$
$$= \frac{w}{2h}\sqrt{a^4 + 4h^2x^2}.$$

128. Exercise. — The Brooklyn Bridge is a suspension bridge, but in order to prevent oscillation from the wind and unsymmetrical loading, the cable has been relieved somewhat by stays and stiffening trusses. The bridge has a span of 1595 feet. The sag OD (see Fig. 79) is 128 feet, the weight per foot supported by each cable is 1200 pounds. Find the terminal and horizontal tensions, assuming that there are no stays or stiffening trusses.

Note. — **The numbers above differ slightly from actual figures.**

129. The Catenary. — If a flexible inextensible cord of uniform density and cross section be suspended from two fixed points A and B, it assumes a position of equilibrium under the action of gravity.

The curve thus formed is the Catenary.

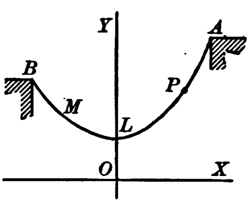


Fig. 81

To find the equation of the curve, let w equal the weight per unit length of the string.

Let s equal length of arc measured from lowest point L to any point P on the catenary.

Evidently the arc (Fig. 81 a) is in equilibrium under the terminal tensions T, acting in a tangential direction at P, and H in a horizontal direction at L, and the weight ws.

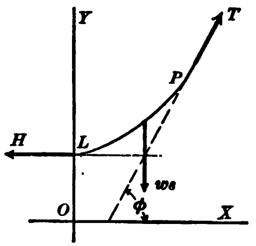


FIG. 81 a

Resolving vertically and horizontally, we have

$$T\cos\phi = H,$$
$$T\sin\phi = ws$$

(since the weight of the arc is ws).

Hence, $$\tan\phi = \frac{ws}{H} = \frac{s}{c},$$

where $c = \frac{H}{w}$, the length of cord whose weight equals the constant horizontal tension. Now,

$$\frac{dy}{dx} = \tan\phi = \frac{s}{c}, \qquad (a)$$

and $$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx; \qquad ds = \sqrt{\frac{c^2 + s^2}{c^2}}dx.$$

$$\therefore \frac{dx}{c} = \frac{ds}{\sqrt{c^2 + s^2}};$$

Integrating,

$$\frac{x}{c} = \log\left[\frac{s + \sqrt{c^2 + s^2}}{c'}\right],$$

where $\log c'$ is a constant of integration.

Take the exponentials of both sides,

$$e^{\frac{x}{c}} = \frac{s + \sqrt{c^2 + s^2}}{c'}.$$

Since now $s = 0$ when $x = 0$, $c' = c$.

Hence, $$c \cdot e^{\frac{x}{c}} = s + \sqrt{c^2 + s^2}.$$

Transpose, square, and solve for s,

$$s = \frac{c}{2}\left[e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right]. \quad \ldots \ldots \ldots \quad (b)$$

Substituting the value of s from (a) in (b) and integrating,

$$y + c'' = \frac{c}{2}\left[e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right],$$

where c'' is a constant of integration.

If we take the origin of coördinates at a distance c below the lowest point of the curve, *i.e.* if $y = c$, when $x = 0$, then $c'' = 0$, and our equation becomes

$$y = \frac{c}{2}\left[e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right], \quad \ldots \ldots \ldots \quad (c)$$

which is the equation of the curve.

Cor. I. Since $c (= OL)$ is the length of the cord whose weight is equal to the horizontal tension, and therefore the tension at the lowest point, it follows that if the part BL of the curve were removed, and a cord of length c of the same weight per unit length as the cord of the curve were joined to the arc LP, and suspended over a smooth peg at L, the curve LPA would still be in equilibrium.

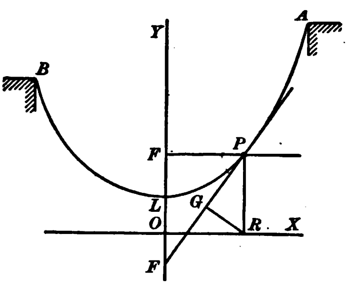


Fig. 82

Cor. II. Since the sides of the triangle PEF are parallel to the three forces under which the arc PL is in equilibrium, it follows that PEF is a force triangle and hence the length of the sides are proportional to the forces. Therefore, we have from the triangle PEF,

$$\frac{\text{Tension at } L}{\text{Tension at } P} = \frac{PE}{PF} = \cos\phi = \frac{dx}{ds}.$$

That is, $\dfrac{wc}{T} = \dfrac{dx}{ds} = \dfrac{c}{y}$ (differentiate (*b*) and substitute in (*c*)).

$$\therefore \; T = yw.$$

That is, the tension at any point of the catenary is equal to the weight of a portion of the cord whose length is equal to the ordinate at that point; that is, any arc P_1P_2 (Fig. 83) will preserve its shape if, instead of being suspended as in Fig. 81, the cord $M_1P_1P_2M_2$ (Fig. 83) is continuous and passes over smooth pegs at P_1 and P_2, and P_1M_1 and P_2M_2 equal respectively the ordinate of the points P_1 and P_2.

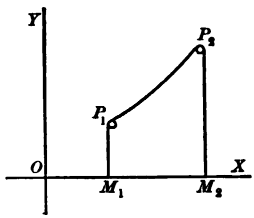


Fig. 83

Cor. III. Subtracting the square of (*b*) from the square of (*c*) we obtain

$$y^2 = s^2 + c^2. \qquad \text{(d)}$$

Cor. IV. If we expand $e^{\frac{x}{c}}$ and $e^{-\frac{x}{c}}$ into a series,

$$y = c\left[1 + \frac{x^2}{c^2\underline{|2}} + \frac{x^4}{c^4\underline{|4}} + \cdots\right] \qquad \text{(e)}$$

130. Near the origin c is large compared with x, that is, $\dfrac{x}{c}$ is a small quantity. Hence, neglecting $\left(\dfrac{x}{c}\right)^4$ and all its higher powers, equation (*e*) becomes

$$y = c\left(1 + \frac{x^2}{c^2\,2}\right).$$

This is the equation of a parabola. That is, for small abscissæ, the ordinates of the catenary are not very different from the ordinates of the parabola; therefore, near its lowest point the catenary approximates in shape a parabola.

131. If we are given the horizontal distance a, from the lowest part of the curve L to one of the supports A, and the

length of the arc $LA = s$, we can find approximate values of c, y, T, H, and $y - c$, the sag of the curve. For

$$s = \frac{c}{2}\left[e^{\frac{a}{c}} - e^{-\frac{a}{c}}\right].$$

Hence, in order to find c it is necessary to solve a transcendental equation. In many practical problems it becomes necessary to solve transcendental equations; for that reason we shall develop here a method for finding an approximate solution of these equations.

132. To solve an Approximate Numerical Equation involving Transcendental Functions.

Suppose $f(z) = 0$ be the given equation, and that we know a value a which differs from z, a solution of $f(z) = 0$, by a small quantity h; then we have by Taylor's Theorem

$$\begin{aligned} f(z) &= f(a + h) \\ &= f(a) + hf'(a) + \frac{h^2}{2} f''(a) + \cdots = 0. \end{aligned}$$

If h is small, we may neglect higher powers of h and get the equation

$$f(a) + hf'(a) = 0.$$

$$\therefore h = -\frac{f(a)}{f'(a)}.$$

$$\therefore z = a - \frac{f(a)}{f(a)} \text{ as the first approximation for } z.$$

Let this new value be called b, and it differs from the true value z by a small quantity $k < h$.

$$\therefore f(z) = f(b + k) = f(b) + kf'(b) + \frac{k^2}{2} f''(b) \cdots = 0.$$

Neglecting higher powers of k, we have an approximate value of k determined from the equation

$$f(b) + kf'(b) = 0.$$

$$\therefore k = -\frac{f(b)}{f'(b)}.$$

$$\therefore z = b - \frac{f(b)}{f'(b)} \text{ as second approximate value of } z.$$

Repeating this process, we may obtain a solution which approximates as nearly as we please the true solution. Usually one or two approximations are sufficient.

To illustrate: A chain 30 feet long is suspended from two points in a horizontal line 20 feet apart. Find c, the sag of the chain, and the terminal tensions.

$$s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right).$$

Hence, at the point of support,

$$\frac{c}{2}\left[e^{\frac{10}{c}} - e^{-\frac{10}{c}}\right] - 15 = 0;$$

or expanding, by Taylor's Theorem, $e^{\frac{10}{c}}$ and $e^{-\frac{10}{c}}$ and collecting, neglecting higher powers than third

$$c\left[\frac{10}{c} + \frac{10^3}{6c^3}\right] - 15 = 0.$$

$$\therefore \frac{10}{c} = \sqrt{3} = 1.732.$$

Let now $$\frac{10}{c} = z.$$

$$3z = e^z - e^{-z}.$$

Let $$f(z) = e^z - e^{-z} - 3z = 0, \text{ and } a = 1.732,$$

then $$h = -\frac{e^{1.732} - e^{-1.732} - 3 \cdot 1.732}{e^{1.732} + e^{-1.732} - 3} = -0.099.$$

$$\therefore b = 1.732 - 0.099 = 1.633.$$

Similarly, $$k = -0.0092.$$

We shall take, therefore,

$$z = \frac{10}{c} = 1.623.$$

$$\therefore c = 6.16.$$

From equation (d),

if y = the ordinate of one of the ends of the chain,

$$y = \sqrt{s^2 + c^2} = 16.22 \text{ feet (nearly).}$$

The sag in the chain $= y - c = 10.06$ feet (nearly).

The terminal tension, $T = 16.22\,w$ lbs. where w = weight per unit length.

133. **Exercises**

1. A chain 62 feet long, weighing 20 pounds per foot, is suspended at two points in a horizontal line 50 feet apart. Find c; the horizontal tension; the terminal tension; the sag of the string.

Ans. $c = 21.55$ ft.; $H = 431$ lbs.; $T = ?$ Sag $= 16.2$ ft.

2. A cable weighing 1 pound per foot, 20 feet long, will withstand a tension of 18 pounds. It is suspended between two points in a horizontal line. What is the greatest distance these points may be apart? *Ans.* 18.76 feet.

134. It is possible to load strings in such a way, or to vary the cross section so that the curve may be any one of a great number, such, for example, as the parabola and catenary, as we have shown, or the ellipse or hyperbola. An important one is the *Catenary of Uniform Strength.* In this curve, the area of the cross section of the string at any point is proportional to the tension at that point. There will then be a constant tension per unit of area of cross section, and the tendency to break will be the same at all points. Students desiring to pursue the subject further will consult Minchin's *Statics*, Vol. I, p. 350.

PART II. KINETICS

CHAPTER IX

THE KINETICS OF A PARTICLE

135. Motion of a Particle in a Straight Line. — We have seen, Newton's First Law of Motion, that if a particle is at rest or is moving in a straight line with uniform speed, the resultant of all the forces acting on it is zero. This case we have considered under the head of *Statics.* There remains then to consider the motion of bodies acted upon by forces which are not in equilibrium. We shall begin with the simplest case, viz. the *motion of a particle in a straight line.* Now, by the Second Law of Motion, a particle will move in a straight line if, and only if, the resultant of all the forces acting on it are directed along the line of motion, and since all the forces acting on a particle may be replaced by their resultant, we shall consider the particle as acted upon by a single force. Let us now consider the problem:

Given the *mass* of a particle, the *force acting on it* and *its position* and *velocity at any time,* to find *its position* and *velocity at any other time.*

136. The *position* of a particle moving in a straight line is known, if its distance and direction from a point O in that line are known. Let the particle be at P, and let the distance of the point P from O be s. If then we assume distances in one direction as positive and in the opposite direction as negative, the algebraic value of s gives the position of P uniquely. It is usual to call distances to the right positive.

FIG. 84

If at a time t_1, P is at A, a distance of s_1 from O, and at a later time t_2 it is at B, a distance s_2 from O, the distance $s_2 - s_1$ is called the *displacement* of P. The displacement is also given a sign. It is usual to give it the positive sign if the displacement is in the direction in which the distance is positive. With this agreement, $s_2 - s_1$ gives both the sign and magnitude of the displacement.

FIG. 85

137. We shall restate the definitions given in Arts. 9 and 10. Let $t_2 - t_1 = \Delta t$ be the time that elapses while the particle makes the displacement $s_2 - s_1 = \Delta s$. If v, the velocity of the particle, is uniform,

$$v = \frac{\Delta s}{\Delta t}.$$

If the velocity is variable and if $\Delta s \doteq 0$ as $\Delta t \doteq 0$, then

$$v = \lim_{\Delta t \doteq 0} \left(\frac{\Delta s}{\Delta t} \right) = \frac{ds}{dt}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If, as usual, we assume Δt as positive, it follows from our agreement regarding the sign of the displacement that the sign of the velocity is the same as the sign of the displacement.

138. Definition. — *Acceleration is the rate of change of velocity.* If the velocity of a particle changes from v_1 to v_2 in the time $t_2 - t_1$, and if we put

$$v_2 - v_1 = \Delta v,$$

and

$$a = \text{acceleration},$$

and if the acceleration is uniform, then

$$a = \frac{\Delta v}{\Delta t}.$$

If the acceleration is not uniform, and if $\Delta v \doteq 0$ as $\Delta t \doteq 0$

$$a = \lim_{\Delta t \doteq 0} \left(\frac{\Delta v}{\Delta t} \right) = \frac{dv}{dt}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$a = \frac{d^2 s}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since $\frac{dv}{dt} = a$, and $\frac{ds}{dt} = v$, it follows that

$$vdv = ads. \quad \quad (34)$$

If v is increasing algebraically, we call the acceleration positive. That is, the acceleration is *positive* if P is moving in a positive direction and the speed is increasing, or if P is moving in a negative direction, and the speed is decreasing. The acceleration is *negative* if P is moving in a negative direction, and the speed is increasing, or if P is moving in a positive direction and the speed is decreasing.

139. Unit of Velocity. Unit of Acceleration. — A particle moving in a straight line with uniform velocity is said to be moving with a *unit velocity* if it describes a *unit space*, in a *unit time*, and with a velocity v, if it describes v units of space in a unit of time. For example, with the choice of units as in Art. 5, the velocity of a particle is 1, if it describes 1 foot in 1 second; and in this case we say the velocity of the particle is 1 foot per second.

If it be moving with a variable velocity, we say that it has a velocity v at any instant, if, moving uniformly for a unit of time, with a velocity v, it would describe v units of space.

A body is said to have a unit acceleration if, with the choice of units as in Art. 5, the velocity changes 1 foot per second in a second.

If the velocity changes a feet per second in a second, we shall say, simply, that it has an acceleration of a feet per second per second. Some authors use for this expression, "an acceleration of a feet per second," others, "an acceleration of $a \frac{\text{ft.}}{\text{sec.}^2}$."

140. The Fundamental Units. Conversion Factors. — We remarked in Art. 2 that every physical equation could be expressed in terms of the units of length, mass, and time. For this reason, they are called the *fundamental units;* and all other

units such as the units of velocity, acceleration, force, momentum, etc., are called *derived* units. The student will observe that *every true physical equation is homogeneous in these fundamental units*, a fact that often enables him to check results.

One of two systems of fundamental units is usually used; viz., the foot-pound-second system, the one which we have employed, or the centimeter-gram-second system, known as the c. g. s. system. The former system is used commercially in English-speaking countries, and by American and English engineers; the latter system, by all countries where the metric system prevails and in practically all work in theoretical physics.

It is often desirable to express quantities given in the units of one system in terms of quantities expressed in the units of some other system; to facilitate which, as well as the checking of results referred to, we shall write below the so-called "dimensional equations." These are not true algebraic equalities, but are intended merely to show the dimensions with which the fundamental units enter into the derived units. For example, $v = \left[\frac{l}{t}\right]$ means only that velocity is of degree 1 in length and degree -1 in time. Similarly,

$$a = \left[\frac{v}{t}\right] = \left[\frac{l}{t^2}\right]$$

means that acceleration is of degree 1 in length and degree -2 in time.

$$\text{Force} = F = [m \cdot a] = \left[\frac{m \cdot l}{t^2}\right]$$

means that force is of degree 1 in mass and in length, and of degree -2 in time.

$$\text{Momentum} = [mv] = \left[\frac{ml}{t}\right].$$

Others will be added as occasion requires.

The *symbolic* expressions in the square brackets are called "change ratios" or "conversion factors," because if we wish to express a quantity given in one system of units in terms of another system of units, we merely need substitute into the conversion factor the ratio of the magnitude of the old unit to the new, and multiply the result by the original quantity.

For example:

A body is traveling with a uniform velocity of 88 feet per second; find its velocity in miles per hour.

The conversion factor is $\left[\frac{l}{t}\right]$,

$$\text{then } \frac{\text{one foot}}{\text{one mile}} = \frac{1}{5280}; \quad \frac{\text{one second}}{\text{one hour}} = \frac{1}{60 \cdot 60}.$$

$$\therefore v = \frac{60 \cdot 60}{5280} \cdot 88 = 60 \text{ miles per hour.}$$

The student will read with profit Everett's: *Units and Physical Constants* or *The Smithsonian Physical Tables*. Care must be exercised that all quantities of one kind used in an equation be expressed in the same units, or at least that the equation be homogeneous in any set of quantities using a given unit.

For example, the acceleration due to gravitation of the earth upon bodies falling freely is usually designated by g. Near the earth's surface $g = 32.2$ feet per second per second (nearly). In an equation in which $g = 32.2$, the unit of time must be chosen as the second, unless the equation is homogeneous in time, and as the unit of space we must choose one foot, unless the equation is homogeneous in lengths.

141. Exercises

1. A body is moving in a straight line with a velocity of 40 miles per hour. Find its velocity in feet per second.

2. Express a velocity of 30 miles an hour in centimeters per second.

3. A body is moving with an acceleration of 32 feet per second per second. Express the acceleration (*a*) in terms of feet per minute per minute; (*b*) in terms of the c. g. s. units.

4. Find the "conversion factor" for changing an acceleration expressed in terms of feet and seconds into centimeters and seconds.

142. Relation between Mass, Force, and Acceleration. — The relation between mass, force, and acceleration is given (Art. 12) by the equation

$$\text{Force} = k \cdot \text{Mass} \cdot \text{Acceleration}, \quad . \quad . \quad . \quad . \quad (5)$$

where k is a constant, depending on the choice of units. The *weight* of a body is the *force* with which the earth attracts it when near the earth's surface. Under this force the body will, if free to fall, be subject to an *acceleration* which we shall denote by g.

Let $W =$ the weight of the body.

$m =$ its mass.

$g =$ acceleration due to gravity.

$F =$ a force which acting on m produces an acceleration a.

Then $$W = kmg,$$

and $$F = kma.$$

$$\therefore F = \frac{W}{g} a. \quad \text{. . . . .} \quad (35)$$

The quantity $\frac{W}{g}$ *is constant.* For, let W_1 be the attraction of the earth for a body and g_1 the acceleration of the body due to this attraction, at one place, and W_2, g_2, the corresponding values when the same body is at another place. Then

$$W_1 = kmg_1,$$

$$W_2 = kmg_2.$$

$$\therefore \frac{W_1}{g_1} = \frac{W_2}{g_2}, \quad \text{. . . . .} \quad (36)$$

which was to be proved.

143. Units of Force and of Mass. — We have defined (Art. 14) as the unit of force, the weight of a unit mass (Art. 3) at sea level, latitude 45°. It is at once evident from equation (36) that since W, the weight of a body, varies (Art. 14), g will be different for different localities. It is also evident that these variations are slight. We shall accordingly assume as a standard g, the acceleration produced by the attraction of the earth at its surface, on a body free to fall, at sea level, latitude 45°.

Substituting $W = 1$, $m = 1$, $a = g$ in the equation,

$$\text{Force} = k \cdot \text{Mass} \cdot \text{Acceleration}$$

we obtain
$$k = \frac{1}{g}.$$

Hence in these units the relation between weight, mass, and acceleration is

$$\text{Weight} = \frac{\text{Mass}}{g} \cdot g,$$

meaning only (what follows from the definition) that the numerical measure of the weight = the numerical measure of the mass.

This system is universally employed commercially and by engineers, and is the one we have adopted and shall use.

For many theoretical investigations it is desirable to have the constant $k = 1$. To effect this, we may let the unit force be defined as a force which produces a unit acceleration on a unit mass. For, substituting

$$F = 1, \quad m = 1, \quad a = 1 \text{ in } F = k \cdot m \cdot a,$$

we find $\quad k = 1.$

Consider, now, a particle of mass m. Let W = number of units of force that represents its weight and let g = the acceleration caused by the force. Then,

$$W = mg;$$

that is, *a unit of force of the first system is g times that of the second system.*

This second unit of force, when the mass is one pound and the unit of length is one foot, is called the *poundal.* If the unit of mass is one gram, and the unit of length one centimeter, the unit of force is called a *dyne.*

No confusion need arise. The equation

$$F = \frac{W}{g} a$$

is true in any system of units, F being expressed in the same units as W. Having found F in terms of units of one system, it can be easily expressed in terms of the units of any other system by using the conversion factor (Art. 140).

In what follows, we shall use the unit of force, as we have hitherto, as the weight of one pound at sea level, latitude 45°.

144. Suggestions as to the Solution of Problems. — The equation (35)

$$F = \frac{W}{g} a = \frac{W}{g} \frac{d^2s}{dt^2}$$

is called the *equation of motion of the particle,* and is simply a mathematical statement of the mechanical conditions given in the problem. The student must remember that F is the resultant of *all* the forces acting on the particle and is directed along its line of motion. For example, suppose a particle lying on a horizontal plane is acted upon by a force P and is subject to a resistance R, such as friction, both forces acting along the line of motion of the particle; then $F = P - R$, since N, the normal reaction of the plane, and W annul each other.

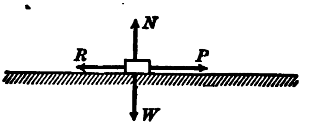


FIG. 86

Having determined F, we merely need to write

$$F = \frac{W}{g} a \text{ or } F = \frac{W}{g} \frac{d^2 s}{dt^2}.$$

We have said (Art. 135) that the problem we have to consider is: having given the forces acting on a particle of given mass, and its position and motion at any time, to find its position and motion at any other time. To do this we *first write the equation of motion.* In most problems occurring in nature F may be expressed in terms of the position of the particle, thus giving an equation which contains the coördinates of the point, and their derivatives with respect to t. This is called a *differential equation.* In order to complete the solution, it is necessary to *integrate this differential equation.*

The integration of differential equations is oftentimes perplexing and difficult, and has given rise to an entirely distinct branch of mathematics, called *Differential Equations.* There are various types of differential equations, *the characteristics of which are determined by the way in which the dependent and independent variables enter into the equation.* Each type requires its own method of integration. In general, we try (1) to express the equations in terms of two variables, and (2) to *separate* the variables, that is, to write the equations so that any one term contains only one variable.

Two types of constants enter into dynamical equations: (1) Those which partake of the nature of a coefficient. For example, in the equation $F = k \cdot ma$, k is a constant of this type. Its value depends upon the choice of the units and must be obtained either by observation or experimentation. (2) The constants of integration. These must be determined by the initial conditions of the problem, and in general are additive.

If the equations of motion involve first derivatives only, they can be solved by *one* integration, and therefore *one* constant of integration is introduced; and therefore only *one* initial condition is assigned. If, however, the acceleration is involved, second derivatives, $\frac{d^2s}{dt^2}$, enter, and therefore it is necessary to integrate *twice*, and hence *two* constants of integration are introduced. To determine these constants *two* initial conditions are required.

To illustrate the preceding remarks, we have, in Arts. 147, 151, and 158, integrated the three different types of differential equations that occur most frequently in problems in dynamics. The student, while reading those articles, will reread this one with profit. We shall add the *résumé:*

In order to solve any problem in Dynamics, we proceed as follows:

(1) *Write the equations of motion.*

These are gotten from the conditions stated in the problem.

(2) *Integrate the equations of motion.*

(3) *Determine the constants of integration*, by a proper substitution of the given initial conditions.

145. Forces that enter into Dynamics.—In a general way forces may be communicated to a particle in the following ways:

(1) By pressure between two particles in contact.

(2) By a flexible, inextensible string; the remarks of Art. 42 apply to this case.

(3) By a flexible elastic string, or by springs,—discussed in Arts. 154 and 155.

(4) By collision with other bodies, discussed in Chap. XIII.

(5) Resistance to motion in gases and liquids. This is a particular case of (4). See Art. 205.

(6) Resistance of friction to the motion of a particle on a rough surface.

As in "static" friction, the *direction* of "kinetic" friction is along the tangent to the surface over which the particle moves and opposite to the direction of its motion. Also, if F is the friction and N the normal reaction between two given bodies, it has been found that (approximately)

$$\frac{F}{N}=f,$$

where f is a constant depending on the relative speed, the nature and the lubrication of the surfaces in contact, and is determined by experiment. In what follows we shall assume that the *magnitude* of the friction is given by the equation $F=f\cdot N$. (See, however, Chap. XVI.)

(7) Attraction of bodies for each other. We shall assume as true the great physical law, known as Newton's Law of Gravitation, namely:

Every particle in nature attracts every other particle with a force directly proportional to the product of their masses, and inversely proportional to the square of the distance between them.

The form of this law and its subsequent observational justification has influenced more profoundly than any other the physical investigations of the past two hundred years.

It will be noted that the law is true for *particles*. It is not true for *bodies* in general. See problem 5, Art. 159. However, it can be shown that any sphere attracts any particle outside the surface of the sphere as if its mass were concentrated at its center. A special case is the attraction of the earth for particles.

(8) Electric and magnetic attraction and repulsion.

146. We shall now illustrate the remarks of Art. 144 by solving a series of problems.

A particle moves in a straight line and its acceleration varies as the time it has been moving. Given the following initial conditions $t = 0$ when $s = s_0$ and $v = v_0$; to find its position and velocity at any time t.

The equation of motion is

$$\frac{d^2s}{dt^2} = kt, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (a)$$

where k is the factor of proportionality. Equation (a) may be written

$$\frac{dv}{dt} = kt.$$

$$\therefore \; dv = ktdt,$$

an equation which contains two variables and in which the variables are separated (see Art. 144).

Integrating, $\quad v = \dfrac{kt^2}{2} + c_1.$

Since from initial condition $v = v_0$, when $t = 0$,

$$c_1 = v_0.$$

We may therefore write

$$v = \frac{ds}{dt} = \frac{kt^2}{2} + v_0; \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (b)$$

$$\therefore \; s = \frac{kt^3}{6} + v_0t + c_2$$

$$= \frac{kt^3}{6} + v_0t + s_0.$$

Eliminating t from (a) and (b) we may find v in terms of the acceleration.

147. A particle whose weight is W moves in a straight line under the action of a constant force acting in the line of motion.

Find the relation between velocity, space, and time, if $s = s_0$, $v = v_0$, when $t = 0$. The equation of motion is

$$\frac{W}{g}a = F; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$$\therefore a = \frac{dv}{dt} = k, \left(k, \text{ a constant} = \frac{Fg}{W}\right);$$

$$\therefore v = kt + c, \text{ (where } c \text{ is a constant of integration).}$$

From the initial conditions, $c = v_0$;

$$\therefore v = kt + v_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

The student may show that

$$s = \frac{kt^2}{2} + v_0t + s_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

This is a relation between the space and the time.

Integrating $vdv = ads$ (equation 34) we get, since $a = k$,

$$\frac{v^2}{2} = ks + c.$$

$$\therefore \frac{v^2}{2} - \frac{v_0^2}{2} = k(s - s_0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

This is a relation between the space and the velocity.

148. Falling Bodies near the Earth's Surface. — An important application of Art. 147 is that of bodies moving vertically, near the surface of the earth, under the action of gravity. The force of attraction of the earth for a given body is approximately constant so long as the body is at the usual distance either above or below the earth's surface. We shall assume, for the present, therefore, that the earth's attraction is constant and also that the motion of the body is not affected by the resistance of the air, which is true only if the body is moving *in vacuo*.

Choosing the surface of the earth as origin and the positive direction upward,

$$k = -g.$$

Then s, the distance of the body *above* the earth's surface at any time, t, is given by

$$s = -\frac{gt^2}{2} + v_0 t + s_0. \qquad (40)$$

If the body is projected from the origin vertically upward with a velocity v_0, then $s_0 = 0$. Hence,

$$v = -gt + v_0;$$

$$s = \frac{-gt^2}{2} + v_0 t. \qquad (41)$$

The body will rise until $v = 0$; therefore the time of the ascent is,

$$t_1 = \frac{v_0}{g}.$$

Substituting this value of t_1 in (41),

$$s_1 = \frac{v_0^2}{2g},$$

where s_1 is the distance the body will rise.

Putting $s = 0$ in (41), and solving for t,

$$t = \frac{2v_0}{g};$$

that is, the time of the ascent equals the time of the descent.

To find the velocity with which a body, initially at rest, reaches the earth if it falls from a point at a distance h above the surface of the earth, we substitute $s = 0$, $v_0 = 0$, $s_0 = h$, and $k = g$ in (39). Therefore

$$v^2 = 2gh. \qquad (42)$$

149. The quantity g is determined by observation. We shall adopt for its value

$$g = 32.2 \text{ feet per second per second.}$$

It is difficult to measure the acceleration produced by gravity from direct observations of a body falling freely, because either the distance fallen must be very great or the time of falling

very small. These difficulties may be obviated somewhat by using *Atwood's Machine.* The principle of this device is as follows:

Two bodies whose weights are respectively W_1 and W_2, $W_1 > W_2$, are connected by a weightless string which passes over a pulley P which turns without friction. The tension is therefore the same in all parts of the string. The acceleration of W_1 downward will be the same as the acceleration of W_2 upward.

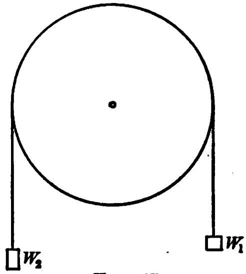


FIG. 87

Let $\quad T =$ tension of the string;

$\quad a =$ acceleration of W_2 positive upward.

The equations of motion of W_2 and W_1 are respectively

$$T - W_2 = \frac{W_2}{g} a;$$

$$W_1 - T = \frac{W_1}{g} a;$$

$$\therefore a = \frac{W_1 - W_2}{W_1 + W_2} \cdot g;$$

$$T = \frac{2\, W_1 W_2}{W_1 + W_2}.$$

Having obtained the value of a by observation, we can calculate g. Substituting in equations (37) and (38), we can find at any time the velocity of the bodies and the space through which they have moved. Take as the origin the initial position of W_2 and suppose the bodies start from rest, then

$$v = at,$$

$$s = \tfrac{1}{2} at^2,$$

$$v^2 = 2\, as.$$

The student may find s and v at any time t, supposing that (*a*) W_2 is projected upward with a velocity v_0; (*b*) that W_2 is projected downward with a velocity v_0.

150. **Exercises**

1. A body moves so that its velocity is positive and varies inversely as the cube of its distance from a given point, O. It is 2 feet from this point when $t = 0$. Find its distance from O at any time.

Ans. $s^4 = 4\,kt + 16$, where k is the factor of proportionality.

2. If the body in exercise 1 move over 1 foot in the first second, find k. How far will it be from the origin in 10 seconds? *Ans.* $k = \frac{65}{4}$; $s = (666)^{\frac{1}{4}}$.

3. A body is projected upward with a velocity of 100 ft. per second. When will it come to rest? How high will it rise?

4. A body is dropped into a well 84 ft. deep. How long before the sound of striking the bottom will be heard? Sound travels 1100 ft. per second. *Ans.* 2.36 sec.

5. A body starting with a velocity of 10 feet per second, and moving with constant acceleration, describes 90 feet in 4 seconds. Find the acceleration.

6. A body starts from rest having a constant acceleration of 25 feet per second per second. In what time will it acquire a velocity of 1000 feet per second?

7. A body is projected upward with a velocity of 80 ft. per second. After what time will it be 20 ft. above the initial position? *Ans.* 0.26 sec.; or 4.72 sec.

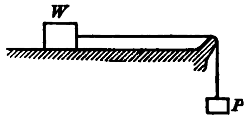
8. A body weighing 100 lbs. rests on the floor of an elevator which is ascending with an acceleration of 16.1 ft. per second per second; what is the pressure on the floor? If it ascends with a uniform velocity, what is the pressure on the floor? *Ans.* 150 lbs. 100 lbs.

9. A loaded railway car on a horizontal track weighs 50 T. The frictional resistance (assumed as constant) to the motion of the car is 1000 lbs. (1) What is the tension on the bar join-

ing it to the engine when the train is being accelerated at a rate of 1 ft. per second per second? (2) What will be the velocity of the train at the end of one minute, if it starts from rest? (3) If the train is moving with a uniform motion, what is the tension on the bar?

Ans. (1) Tension in bar = 4100 lbs. (nearly); (2) velocity = 60 ft. per sec.; (3) tension in bar = 1000 lbs.

10. A body whose weight is W lies on a smooth horizontal table. A weight P is connected to W by a weightless string passing over a smooth pulley at A. (*a*) Find the acceleration of W along the plane; the tension T. (*b*) If W starts from rest, find its velocity at the end of the first second, and the distance it will move in two seconds.



Ans. (*a*) Acceleration $= \dfrac{Pg}{P+W}$, tension $= \dfrac{PW}{P+W}$.

151. Simple Harmonic Motion. — Thus far we have considered the rectilinear motion of a particle, acted upon by a constant force. We shall consider now some of the simpler problems which occur in nature in which the magnitude of the force acting on the particle is variable, the line of action of the force coinciding with the path of the particle. The simplest is the case in which the *magnitude* of the *force* varies directly as the *distance* of the particle from a *fixed point* in the line of motion. Let us now find the *position* and the *velocity* of the particle at any time.

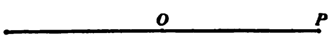


Fig. 88

Let O, the fixed point, be the origin, and P the position of the particle at any time, and $OP = s$. Assume as initial conditions

$$s = s_0,\ v = v_0, \text{ when } t = 0.$$

Then the equation of motion is

$$\frac{W}{g}a = k_1 s,$$

where k_1 is a factor of proportionality. In most problems in nature in which this law applies, the force acts toward the point O, so that when s is positive the acceleration is negative, and when s is negative, the acceleration is positive. Hence dividing through by $\frac{W}{g}$ and putting $\frac{k_1 g}{W} = -k^2$, we obtain

$$a = \frac{d^2s}{dt^2} = -k^2s. \qquad (43)$$

Now $\quad vdv = ads.$

Hence $\quad vdv = -k^2sds;$

$$\therefore v^2 = -k^2s^2 + c_1.$$

But $\quad c_1 = v_0^2 + k^2s_0^2;$

$$\therefore v^2 = \left(\frac{ds}{dt}\right)^2 = v_0^2 + k^2(s_0^2 - s^2). \qquad (a)$$

In most problems that the student will solve, in which this law is involved, the body will start from rest; that is, $v_0 = 0$, when $s = s_0$.

Then $\quad \left(\frac{ds}{dt}\right)^2 = k^2(s_0^2 - s^2).$

$$\therefore \frac{ds}{\pm\sqrt{s_0^2 - s^2}} = kdt. \qquad (44)$$

Choosing the negative sign of the radical and integrating, we obtain,

$$\cos^{-1}\left(\frac{s}{s_0}\right) = kt + c_2.$$

But $\quad c_2 = 0;$

$$\therefore s = s_0 \cos kt. \qquad (45)$$

The student may prove that if he choose the positive sign of the radical in (44) he will also obtain (45).

152. Discussion of Equations (44) and (45).—Equation (44) may be written

$$\frac{ds}{dt} = v = k\sqrt{s_0^2 - s^2}, \qquad (44')$$

which expresses the velocity in terms of the distance of the particle from O.

Differentiating (45) we get,

$$\frac{ds}{dt} = v = -ks_0 \sin kt, \quad \ldots \ldots \ldots \quad (46)$$

which expresses the velocity in terms of the time that the particle has been moving.

From (44′) the velocity increases as s decreases, is greatest when $s = 0$, and is equal to zero when, and only when, $s = \pm s_0$ and imaginary when $s^2 > s_0^2$; that is, the particle is never farther from the origin than s_0. For a given distance, s, from the origin, the body has the same velocity, no matter which way it is going.

From (45) we see that $s = s_0$, when

$t = 0, \frac{2\pi}{k}, \frac{4\pi}{k}, \ldots \frac{2n\pi}{k}$, where n is an integer; that is, that the particle will return to the starting point in a time, $t = \frac{2\pi}{k}$.

Moreover, since

$$\cos kt = \cos (kt \pm 2n\pi),$$

we see that if the particle at any time t_1 is at a distance s from the origin, it will be at the same point at any time t, given by the equation $kt = kt_1 \mp 2n\pi$, and as appears from (46) it will be traveling in the same direction.

The motion defined by either equation,

$$s = s_0 \cos kt, \quad \ldots \ldots \ldots \quad (45)$$

or

$$a = \frac{d^2s}{dt^2} = -k^2 s \quad \ldots \ldots \ldots \quad (43)$$

is called *simple harmonic motion*. The quantity $\frac{2\pi}{k}$ is called its *period*, and the greatest departure of the particle from the origin, s_0, is called its *amplitude*. It will be noted that the period does not depend on its amplitude. Also, from (45) when $s = 0$,

$t = \frac{\pi}{2k}$, which is one fourth of a period. Hence the particle starting from rest from the position s_0, will reach O in the same time, whatever value s_0 may have.

REMARK. — The reader will note the *form* of equation (43). On one side of the equation is the second derivative of the dependent variable s, with respect to the independent variable t; on the other side occurs the first power of the dependent variable, multiplied by a *negative* constant, $-k^2$. If further the first derivative $\frac{ds}{dt} = 0$ when $t = 0$ and $s = s_0$, the integral of this equation is

$$s = s_0 \cos kt. \qquad (45)$$

This is one of the standard forms of differential equations, and is a very important one.

153. Hooke's Law. — Application of this law of force is found in the study of the mechanics of elastic strings, the extension and compression of bridge members under moving loads, the compression and extension of spiral springs, and the deflection of beams. In applying it we assume Hooke's Law.

HOOKE'S LAW. — *The tension is proportional to the extension* of an elastic bar, spring, or string. The law is not strictly true, but is so nearly true when the extension is small compared with the natural length of the bar or string that it is always assumed in the class of problems named in this article. It is common knowledge, that if we stretch a string or bar beyond a certain limit, that it (the string or bar) will not regain its natural length. In this case Hooke's Law does not hold even approximately.

The same law holds true for the *compression* of an elastic bar or spring. That is, the force with which a bar or spring resists shortening varies as the amount that the body is compressed.

154. Let us consider the following example.

Let a body B of weight W lie on a smooth horizontal table, and be attached to a fixed point O in the table by an

elastic weightless string whose natural length is l. Let the body B be displaced so that the string is extended beyond its natural length, and then released. Determine the subsequent motion of the body. Since W and N (Fig. 89) annul each other, the only force which tends to move the body after its release is T, the tension of the string, which is directed toward O.

FIG. 89

Let $s_0 =$ the greatest length of the string.
$s =$ the length of the string at any time, t.
$T =$ the tension at any time.

Then, from Hooke's Law,

$$T = k(s - l). \quad \ldots \ldots \quad (a)$$

Like all factors of proportionality, the value of k may be found by experiment. It is usual to determine it thus: Removing the mass from the table, let it hang freely, supported by the string, which will now be stretched to a length l_1 (say), and for this extension let $T = T_1$.

But $\qquad T_1 =$ the weight of body $= W$.

$$\therefore k = \frac{W}{l_1 - l}.$$

Substituting this value of k in (a),

$$T = \frac{W}{l_1 - l}(s - l).$$

Let us put $\qquad l_1 - l = e. \quad \ldots \ldots \quad (b)$

Then since T is the resultant force, we have

$$T = -\frac{W}{g}a = \frac{W}{e}(s - l);$$

$$\therefore a = \frac{d^2s}{dt^2} = -\frac{g}{e}(s - l).$$

Now because l is constant,

$$\frac{d^2s}{dt^2}=\frac{d^2(s-l)}{dt^2}.$$

Hence the equation of motion may be written,

$$\frac{d^2(s-l)}{dt^2}=\frac{-g}{e}(s-l). \quad \text{. . . . .} \quad (c)$$

If we think of $(s-l)$ as the dependent variable in this equation, it will be seen that it is of the form of equation (43), Art. 151. Since the particle is pulled away from O until the string is stretched to a length s_0, and then released $s-l=s_0-l$, when $v=0$ and $t=0$.

Hence by equation (45),

$$s=l+(s_0-l)\cos\sqrt{\frac{g}{e}}\,t. \quad \text{. . . . . . .} \quad (d)$$

To find its velocity at any time differentiate (d) with regard to t;

$$\therefore\ v=\frac{ds}{dt}=-\sqrt{\frac{g}{e}}\,(s_0-l)\sin\sqrt{\frac{g}{e}}\,t.$$

It will be noted that the *mechanical* conditions given in the problem cease when $s=l$. At this point,

$t=\frac{\pi}{2}\sqrt{\frac{e}{g}}$; and $v=-(s_0-l)\sqrt{\frac{g}{e}}$. The body will then move with the uniform velocity $v=(s_0-l)\sqrt{\frac{g}{e}}$ until it travels a distance $=2\,l$, when the mechanical conditions are just the reverse of those stated at the beginning of the problem.

155. A kindred problem, and one of much greater practical importance, is the following:

A body of weight, W, is suspended vertically by means of a weightless elastic string. Suppose the body is depressed below the position of equilibrium when W hangs freely and then set at liberty. Determine the subsequent motion.

Let l, l_1, e, T, t, s, and s_0 have the same significance as in the preceding article. The student may prove

1. $T = W\dfrac{(s-l)}{e}$.

2. The equation of motion is

$$a = \frac{d^2s}{dt^2} = -\frac{g(s-l_1)}{e},$$

or
$$\frac{d^2(s-l_1)}{dt^2} = -\frac{g(s-l_1)}{e}.$$

3. $\therefore s - l_1 = (s_0 - l_1)\cos\sqrt{\dfrac{g}{e}}\,t.$

4. $v = \dfrac{ds}{dt} = -\sqrt{\dfrac{g}{e}}(s_0 - l_1)\sin\sqrt{\dfrac{g}{e}}\,t.$

5. Let us now seek the conditions under which, during the motion of the body, the string may regain its natural length, that is, *s becomes equal to l.*

From 3 and 1,

$$T = \frac{W}{e}(s-l) = \frac{W}{e}(l_1 - l) + \frac{W}{e}(s_0 - l_1)\cos\sqrt{\frac{g}{e}}\,t; \quad . \quad . \quad . \quad (e)$$

hence when $s = l$, T vanishes.

$$\therefore \cos\sqrt{\frac{g}{e}}\,t = -\frac{l_1 - l}{s_0 - l_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

and since there is no real angle whose cosine is greater than unity, this equation is impossible if

$$\frac{l_1 - l}{s_0 - l_1} > 1, \quad \text{or } l_1 > \frac{s_0 + l}{2}.$$

That is, the tension can never vanish if

$$l_1 > \frac{s_0 + l}{2}.$$

In this case we find from 4 that

$$v = 0, \text{ when } t = \pi\sqrt{\frac{e}{g}}.$$

The corresponding value of s is

$$s = l_1 - (s_0 - l_1).$$

That is, the particle will rise as high above the position that it assumes when it hangs freely as it was depressed below that position at the beginning of motion. The particle will oscillate through this position with a harmonic motion with a period $= 2\pi\sqrt{\frac{e}{g}}$, and an amplitude $= l_1 - s_0$.

If, however, $\frac{l_1 - l}{s_0 - l_1} \leqq 1$ and hence $l_1 \leqq \frac{s_0 + l}{2}$, then there is a real angle satisfying equation (f); *i.e.*, s may take the value l, at which point T will vanish.

Substituting the value of $s = l$ in 3, we get the corresponding value of t, and substituting this value in 4, find the velocity. The mechanical conditions no longer exist when $s \leqq l$.

Query. — What will the particle do under the following conditions? (1) The velocity is zero when $s = l$; that is, when the string regains its natural length. (2) The velocity is not zero, when $s = l$.

There are many interesting and practical problems intimately connected with the preceding. For example: If the body is dropped from rest from an initial position s_0, where $l < s_0 < l_1$, it will at first move downward, coming to rest again at a point below the position of equilibrium when the body hangs freely, such that

$$s - l_1 = l_1 - s_0$$

and the body will oscillate through this position of equilibrium.

If an additional force act on the body, acting only in the direction in which the body is moving, the amplitude will increase; for example, if the oscillation is vertical and a force act downward, only when the body is moving downward, or upward when the body is moving upward, or both, the amplitude will increase, and the extension will, if the force persists, by and by pass the elastic limit and the string or rod will not regain its natural length. Such a vibration is called a *forced*

vibration. In case the additional forces are always acting in a direction opposite to that of the motion of a particle, the amplitude will decrease. Such a vibration is called a *damped* vibration.

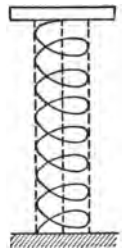


FIG. 90

The force exerted by a spiral spring is another example of a force whose magnitude varies as the distance from a fixed point. In this case the fixed point is the natural position of the head of the spring. The spring differs from the string in that it offers resistance to either extension or compression beyond its natural length, while the string resists extension only.

156. Exercises

1. If the particle moves as described in Art. 151, and $k = 1$, with what velocity will it reach the origin, provided it starts from rest at a distance of 10 feet from the origin? How long will it take it to reach the origin? Where will it be at the end of the first half second?

Ans. $v = 10$ ft. per sec.; $t = 1.5708$ sec.; the particle will be 8.776 ft. on the positive side of the origin.

2. If there is no obstacle in its path, what will the particle do after it reaches the origin? Will it ever stop? Will it ever return to the place from which it started? If so, when?

3. A string whose natural length is 12 feet is stretched one inch by a suspended weight of 5 pounds. If the body be placed on a smooth horizontal table, as in Art. 154, and the string be extended 3 inches, find the velocity with which the particle reaches O. *Ans.* $v = 3\sqrt{g}$ ft. per sec.

4. If, instead of a 5-pound weight, a 10-pound weight were placed on the horizontal table attached to the same string, the other conditions remaining the same, with what velocity would it reach the origin O? *Ans.* $v = \dfrac{3\sqrt{g}}{\sqrt{2}}$ ft. per sec.

5. A body weighing 10 pounds when suspended by a weightless elastic string, whose natural length is 5 feet, extends the string to 6 feet. The 10-pound weight is removed, and another weighing 5 pounds is suspended from the same string. If the 5-pound weight is depressed below the position of equilibrium, and then let go, write the equation of motion.

6. Find how far the 5-pound weight of the preceding exercise would have to be depressed so that the string would just regain its natural length (*i.e.*, tension become zero). Find the period and the amplitude of the motion.

157. Forces varying inversely as the Square of the Distance. — Another law which occurs very frequently in nature is that in which the magnitude of the force varies inversely as the square of the distance of the particle from a fixed center, O. (See Art. 145, 7.) In fact a sphere attracts a particle without the sphere as if the entire mass of the sphere were concen-

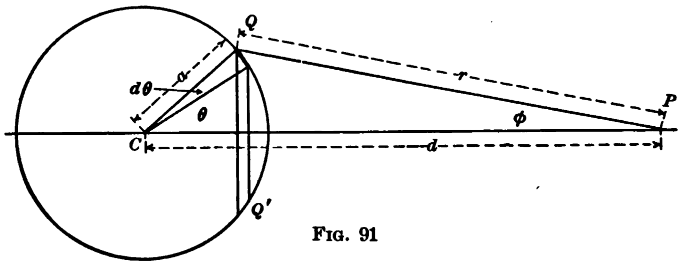


FIG. 91

trated at its center. To prove this, let us find the attraction of a homogeneous spherical shell for a particle.

If the circle in the figure be revolved about the axis PC through an angle 2π, it will generate a thin spherical shell, and the arc $ds = ad\theta$ at Q, included between two infinitely near ordinates, will generate a differential volume whose mass is

$$dm = \frac{\rho}{g} 2\pi a \cdot \sin\theta \cdot ad\theta,$$

where ρ is the weight per unit volume.

The attraction of the arc ds for a particle of unit mass at P is

$$\text{Attraction} = \frac{k\rho a d\theta}{r^2 g}.$$

This attraction may be resolved into two parts, the one = X (say) along PC and the other = Y perpendicular to PC. Now to every particle at Q there is a corresponding particle at Q', such that its attraction for a particle at P along $PC = X$, and the attraction perpendicular to $PC = - Y$, which annuls Y. Hence the entire attraction of dm is along PC, and is equal to

$$dA = \frac{k\rho\, 2\,\pi a \cdot \sin\theta \cdot a d\theta}{gr^2} \cos\phi. \quad . \quad . \quad . \quad (a)$$

From the geometry of the figure,

$$r^2 = a^2 + d^2 - 2\,ad \cdot \cos\theta.$$

$$\therefore r \cdot dr = ad \cdot \sin\theta \cdot d\theta \cdot$$

Also $$\cos\phi = \frac{d - a \cdot \cos\theta}{r}.$$

Substituting these values in (a) we obtain the attraction

$$dA = \frac{k\pi\rho}{g} \frac{a}{d^2}\left(\frac{d^2 - a^2 + r^2}{r^2}\right) dr.$$

Integrating this expression between the proper limits we obtain the total attraction of all such elementary surfaces. There are two cases to be considered.

1. The particle at P is without the shell. The limits of integration in this case are $d - a$ and $d + a$. Hence the total attraction A is

$$A = \frac{k\rho\pi}{g} \frac{a}{d^2}\left[\frac{a^2 - d^2}{r} + r\right]_{d-a}^{d+a} = \frac{4\,\rho\pi}{g} k \frac{a^2}{d^2} = \frac{kM}{d^2}.$$

That is, the attraction of a homogeneous spherical shell for a particle without it is the same as if the mass of the entire shell were concentrated at its center.

COROLLARY. *A sphere may be regarded as made up of a series of concentric spherical shells each of which is of uniform density. If the masses of the shells are m_1, m_2, $\cdots$ respectively, it is evident that the attraction of the sphere is*

$$\frac{k(m_1 + m_2 \cdots)}{d^2} = \frac{k \text{ mass of the sphere}}{d^2}.$$

That is, a sphere attracts a particle without the sphere as if the mass of the sphere were concentrated at its center. This law holds almost exactly, for bodies slightly flattened at the poles, if the particle is not too close to the attracting body. Since both these conditions prevail in the solar system, this law is of much importance in astronomical investigations.

2. The particle is within the shell. In this case the limits are, $a - d$ and $a + d$. Hence

$$A = \frac{\pi k \rho}{g} \frac{a}{d^2} \left[\frac{a^2 - d^2}{r} + r \right]_{a-d}^{a+d} = 0.$$

That is, the resultant of all the attractions of the particles of a spherical shell on a particle within the shell is zero.

158. A Particle moving vertically under the Attraction of the Earth. — We shall assume in this problem that the earth is spherical and that the particle is so far above the surface that the attraction is not constant, but varies inversely as the square of the distance of the particle from the center of the earth. Take the center of the earth, O, as origin, P as the position of a particle whose weight is W; the distance OP as *positive* and equal s; then the force, F, acting on P is toward O, and in magnitude equals $\frac{k_1^2}{s^2}$ where k_1^2 is a constant; that is,

$$F = \frac{k_1^2}{s^2}.$$

To determine k_1^2, we know that when P is at the surface of the earth $F = W$, the weight of P, and $s = R$, the radius of the earth.

$$\therefore k_1^2 = WR^2.$$

Hence the equation of motion is

$$\frac{WR^2}{s^2} = -\frac{W}{g}a. \quad \text{(a)}$$

Substituting for a from $vdv = ads$, equation (a) becomes

$$\frac{-gR^2}{s^2}ds = vdv.$$

Putting $gR^2 = k^2$ and integrating, we obtain

$$v^2 = \frac{2k^2}{s} + c_1.$$

But, if $v = v_0$ when $s = s_0$,

$$c_1 = v_0^2 - \frac{2k^2}{s_0};$$

$$\therefore v = \frac{ds}{dt} = \pm\sqrt{\frac{2k^2}{s} + v_0^2 - \frac{2k^2}{s_0}}.$$

Since s is assumed positive, v can become zero if, and only if, $v_0^2 - \dfrac{2k^2}{s_0}$ is negative. If $v_0^2 - \dfrac{2k^2}{s_0}$ is positive, the velocity will decrease as s becomes larger and become equal $\sqrt{v_0^2 - \dfrac{2k^2}{s_0}}$ when s becomes infinity.

Suppose now that there is a value of $s = s_1$ (say), for which $v = 0$, and that the particle is in this position. It will at once start toward the origin with a negative velocity.

Put $v_0^2 - \dfrac{2k^2}{s_0} = -\dfrac{k^2}{h^2}$ (since $v_0^2 - \dfrac{2k^2}{s_0}$ is negative).

Then $s_1 = 2h^2$.

$$\therefore -\frac{sds}{\sqrt{2h^2s - s^2}} = \frac{k}{h}dt.$$

Integrating both sides and substituting the limits s_1 and s, we obtain

$$\frac{k}{h}t = \left[\sqrt{s_1s - s^2} - \frac{s_1}{2}\operatorname{vers}^{-1}\frac{2s}{s_1} + \frac{s_1\pi}{2}\right]. \quad (47)$$

Giving s any value, we obtain the time for the particle to reach the corresponding position.

If the particle is moving from the origin, the time t required for it to reach the position s is

$$\frac{k}{h}t = \frac{s_1}{2} \text{ vers}^{-1}\frac{2s}{s_1} - \sqrt{s_1 s - s^2}.$$

159. Exercises

1. Assuming that R, the radius of the earth, is 4000 miles, and that $g = 32.2$, find the velocity with which a body falling from rest, from a distance one mile above the surface of the earth, strikes the surface of the earth, neglecting the resistance of the air.

Ans. 583 feet per second (nearly).

2. Find the velocity with which a particle would have to be ejected from the surface of the earth in order that it should never return.

Ans. 6.97 miles per second (nearly).

NOTE. — Such a velocity is called the *critical velocity* or velocity of escape. It is of interest because it is supposed that certain particles of our atmosphere attain this velocity, and therefore escape from the control of the earth.

3. Find the attraction of an infinitely thin homogeneous circular plate of radius a for a particle situated on an axis passing through the center of the plate and perpendicular to its plane at a distance h from the center of the plate.

Ans. If 2θ is the vertical angle of the cone subtended at the particle by the plate and R the total attraction,

$$R = 4\pi k \sin^2\frac{\theta}{2}.$$

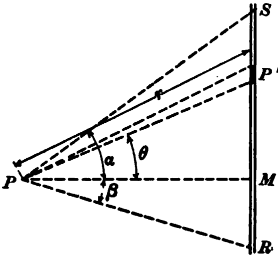
4. Find the attraction of a right circular homogeneous cylinder for a particle on the axis of the cylinder and a distance h from one end.

5. Find the magnitude and direction of the attraction of a thin homogeneous straight rod RS for a particle at P.

Hint. — Choose any element ds located at P', a distance r from P. Let PM be the perpendicular let fall from P on RS. Let $PM = p$.

Then the attraction dX of the element ds for P is

$$dX = k\rho \frac{ds}{r^2} \cos\theta. \quad . \quad . \quad (a)$$



But $\quad s = MP' = p \cdot \tan\theta.$

$$\therefore\ ds = p \cdot \sec^2\theta \cdot d\theta,$$

also $r \cdot \cos\theta = p.$

Substituting in (a) and integrating, the student will prove that the total horizontal attraction X is

$$X = \frac{k\rho}{p}(\sin\beta + \sin\alpha).$$

Similarly, if Y equal the total vertical attraction, the student will prove that

$$Y = \frac{k\rho}{p}(\cos\beta - \cos\alpha).$$

Then to find the total attraction A, use the equation $A = \sqrt{X^2 + Y^2}$ and prove

$$A = \frac{2\,k\rho}{p}\sin\left(\frac{\alpha+\beta}{2}\right).$$

Having found A, prove that the direction of A bisects the angle RPS.

Note. — It will be observed that this attraction is not toward the center of gravity of RS. Indeed it is only in exceptional cases, some of which we have chosen in Art. 157, and the exercises 3 and 4 of this article, that the attraction is toward the center of gravity of the body.

160. Motion in Resisting Medium. — The laws of resistance of gases and liquids to motion of bodies in them are not completely understood; the resistance evidently is some function

of the velocity. We shall assume that the resistance $= k_1^2v^2$, where k_1 is a constant depending on the particular medium.

One of the most important cases is that of a body projected vertically upward from the surface of the earth, subject to the action of gravity, which in this case we shall assume is constant, and to the resistance of the air.

Let the point of projection be the origin, and let the weight of the particle be W, its velocity $v = v_0$ when $s = 0$ and $t = 0$. Since the forces acting on the particle are its weight W, and the resistance of the air $= k_1^2v^2$, we may write

$$\frac{W}{g}a = -(W + k_1^2v^2); \text{ whence}$$

$$a = \frac{dv}{dt} = -g(1 + k^2v^2) \text{ where } k^2 = \frac{k_1^2}{W}.$$

$$\therefore \frac{dv}{1 + k^2v^2} = -gdt;$$

$$\frac{1}{k}\tan^{-1} kv = -gt + c_1;$$

$$c_1 = \frac{1}{k}\tan^{-1} kv_0;$$

$$\tan^{-1} kv - \tan^{-1} kv_0 = -kgt.$$

$$v = \frac{ds}{dt} = \frac{1}{k}\frac{kv_0 - \tan kgt}{1 + kv_0 \tan kgt}.$$

Writing $\tan kgt$ in terms of $\sin kgt$ and $\cos kgt$ and integrating,

$$s = \frac{1}{k^2g}\log(kv_0 \sin kgt + \cos kgt).$$

Exercise 1. — A particle falls from rest subject to the action of gravity and the resistance of the air. Suppose $s = s_0$ when $v = 0$ and $t = 0$.

(*a*) Write the equation of motion of the particle.

(*b*) Find its velocity at any time.

(*c*) Find the velocity with which it strikes the earth.

Note. — **Let the answer involve** k.

CHAPTER X

THE MOTION OF A PARTICLE IN A PLANE CURVE

161. Speed and Velocity.— We have hitherto considered the motion of a particle in a straight line. Its velocity was therefore constant in direction. However, if a body move in a curve, not only the *speed*, which is the numerical value of the velocity, may change, but the direction also changes. For example, suppose a particle describe a circle with uniform speed, in the sense indicated by the arrow. Since the particle at any instant is moving along the tangent to its path, it is evident that the direction of its velocity at P_1 differs from the direction at P_2, and that in this case, although the speed is constant, the velocity is variable, for its direction changes. In many cases both speed and velocity change. Hence, to define a velocity completely, we must give

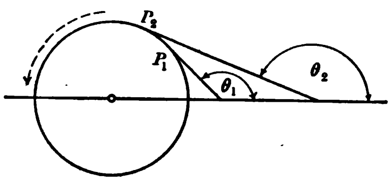


FIG. 92

1. Its magnitude.
2. Its direction.
3. Its sense.

Therefore, we may represent a velocity completely by the segment of a straight line (see Art. 15), which is parallel to the velocity, whose length equals the magnitude of the velocity, and to which is affixed an arrowhead to show its sense.

162. Parallelogram of Velocities. — Let a particle move along a rod with uniform speed which carries it in a unit of time from

O to A; if, at the same time, the rod be carried parallel to itself so that O moves with uniform velocity along OB to B in a unit of time, then it is evident that the particle will be at C at the end of a unit of time. Moreover, at any time during the motion it will be on the diagonal OC; since, at the end of

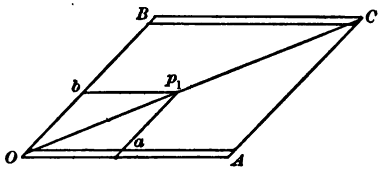


FIG. 93

$\frac{1}{m}$th of a unit of time the particle will have moved a distance Oa along the rod, and O will have moved to b, where Oa and Ob are defined by

$$Oa = \frac{1}{m} OA,$$

$$Ob = \frac{1}{m} OB.$$

$$\therefore \frac{Oa}{Ob} = \frac{OA}{OB},$$

which shows that P_1, the position of the particle at the end of $\frac{1}{m}$th of a unit of time, is on the diagonal OC.

In the foregoing we have tacitly assumed that in carrying the rod we have in no way affected the velocity of the particle *along* the rod — not a violent assumption.

Accordingly we have the important theorem known as

THE LAW OF PARALLELOGRAM OF VELOCITIES. — *If a particle is subject to two simultaneous velocities represented in magnitude and direction by two straight lines OA and OB, it will move as if subject to a single velocity represented in magnitude and direction by the line OC, the diagonal of a parallelogram of which OA and OB are adjacent sides.*

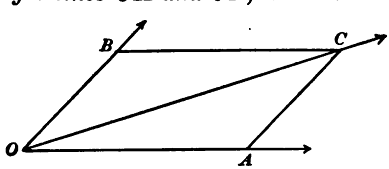


FIG. 94

Conversely, a body subject to a velocity represented in magnitude and direction by the line OC will move in the same way when it is subject to two simultaneous velocities represented by two sides of a parallelogram of which OC is a diagonal. Hence velocity is a vector. (See Art. 68.)

We shall call the velocity represented by OC the ***resultant*** of the velocities represented by OA and OB respectively. The velocities represented by OA and OB are called the ***components*** of the velocity represented by OC. It is evident that we may find, by repeating the process of this article, the resultant of any number of simultaneous velocities. The process of finding the resultant of two or more simultaneous velocities is called the ***composition*** of velocities, and the process of finding the components of a given velocity is called the ***resolution*** of velocities. The case most commonly used is that in which the angle $AOB = 90°$. In this case OA is called the ***resolved part of OC along OX***, and OB is called the ***resolved part of OC along OY***.

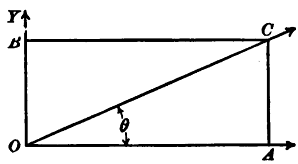


FIG. 95

From the geometry of the figure

$$OA = OC \cos \theta;$$
$$OB = OC \sin \theta.$$

That is, to find the resolved part of a velocity along a line, we multiply the velocity by the cosine of the ***angle between the direction of the velocity*** and the line.

163. We have (in Arts. 9 and 137) defined velocity by the equation

$$v = \lim_{\Delta t \doteq 0} \left(\frac{\Delta s}{\Delta t}\right) = \frac{ds}{dt}. \qquad (2)$$

It must be remembered in using this equation, except when the particle describes a straight line, as in Chapter IX, that Δs involves, not only the length of path described in the time Δt, but its direction also. We might treat velocity as a vector. In what follows, however, we have adopted the methods of the

elementary calculus, giving respectively the magnitude and direction of the velocity of the particle by the two equations

$$\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

$$\tan\phi=\frac{dy}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

The angle ϕ is the angle that the tangent to the path makes with the x-axis at the point where the speed is $\frac{ds}{dt}$.

Equation (48) is derived from the well-known equation $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$, and assumes that each of the coördinates of the point is a function of the time. Hence (48) and (49) give the magnitude and direction of the velocity at any time.

164. Suppose a point P, coördinates x, y, be moving in a curve with a velocity $\frac{ds}{dt}$ and that the tangent to the curve at P makes an angle ϕ with the x-axis. Then from the calculus

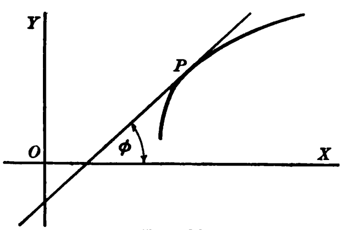


FIG. 96

$$\cos\phi=\frac{dx}{ds},$$

and

$$\sin\phi=\frac{dy}{ds},$$

and

$$\frac{dx}{dt}=\frac{ds}{dt}\cdot\frac{dx}{ds}.$$

Designating by v_x and v_y *the resolved parts of* the velocity along the x- and y-axis respectively, we have (Art. 162)

$$v_x=\frac{ds}{dt}\cos\phi=\frac{ds}{dt}\cdot\frac{dx}{ds}=\frac{dx}{dt};$$

$$\therefore v_x=\frac{dx}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

$$v_y = \frac{ds}{dt}\sin\phi = \frac{ds}{dt}\cdot\frac{dy}{ds} = \frac{dy}{dt};$$

$$\therefore v_y = \frac{dy}{dt}. \quad \text{(51)}$$

165. The Parallelogram of Accelerations. — By Newton's Laws of Motion, a particle is subject to an acceleration if, and only if, it is acted upon by some unbalanced force; and the acceleration of the particle is in the direction of the line of action of this force and proportional to it. Hence we may represent the acceleration produced on a particle by a force P by the same line that represents the force P. Also if the particle is subject to the action of a force Q, we can represent the acceleration produced on the particle by Q by the same line that represents Q. But the particle, by the parallelogram of forces, will behave in the same way when subject to the action of R alone (the resultant of P and Q), as when acted upon by P and Q simultaneously. Hence, the resultant of the two accelerations can be represented by the line that represents R; that is, by the diagonal of the parallelogram, two of the sides of which represent P and Q. It is evident that instead of taking the entire lengths of the sides and diagonal of the parallelogram representing the forces, we may take any proportional parts of them.

Hence from the Second Law of Motion and the law of parallelogram of forces, it follows that if *a particle is subject to two simultaneous accelerations represented in magnitude and direction by two straight lines OA and OB, respectively, the resultant acceleration is represented in magnitude and direction by the diagonal OC, of the parallelogram constructed on OA and OB as adjacent sides.* Hence acceleration is a vector. This is known as the law of *composition of accelerations.* It also follows that any acceleration may be resolved into two or more accelerations. In fact,

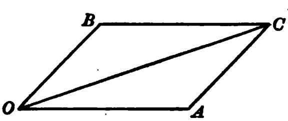


FIG. 97

we may formulate a theorem for acceleration out of each law of composition of forces or resolution of forces, directly dependent for its proof on the parallelogram law by changing the word "force" into "acceleration" in the theorem of Art. 17 *et seq.* For example, there is a theorem of the "polygon of accelerations," analogous to the "polygon of forces," etc. (See Art. 78.)

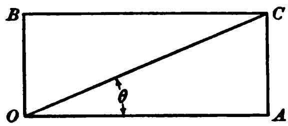


FIG. 98

The case used most frequently is that in which the angle AOB is a right angle. Let a = acceleration along OC. Then the acceleration along $OA = a \cos \theta$, and the acceleration along $OB = a \sin \theta$. We call $a \cos \theta$ the ***resolved part*** of a along OA.

166. Acceleration along the Axes of Coördinates. — Let us put

$$a \cos \theta = a_x = \text{acceleration parallel to the } x\text{-axis},$$
$$a \sin \theta = a_y = \text{acceleration parallel to the } y\text{-axis}.$$

Then

$$a = \sqrt{a_x^2 + a_y^2}. \quad . \quad . \quad (52)$$

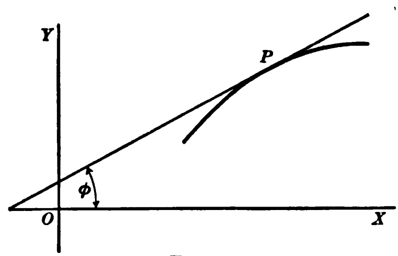


FIG. 99

If a particle at P is moving with a velocity $\frac{ds}{dt}$, then the velocity along the x-axis is (Art. 164)

$$v_x = \frac{dx}{dt}.$$

Since the *x-axis is fixed in direction,* it follows (Art. 138) that

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Similarly,

$$a_y = \frac{d^2y}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

167. Acceleration along the Tangent; along the Normal. — Let us now resolve a_x and a_y along the tangent to the curve, denoting the acceleration along the tangent a_t; then

$$a_t = a_x \cos\phi + a_y \sin\phi = \frac{d^2x}{dt^2}\cdot\frac{dx}{ds} + \frac{d^2y}{dt^2}\frac{dy}{ds}. \quad \text{(See Art. 165.)}$$

Differentiating the well-known equation

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2,$$

we obtain

$$\frac{ds}{dt}\frac{d^2s}{dt^2} = \frac{d^2x}{dt^2}\cdot\frac{dx}{dt} + \frac{d^2y}{dt^2}\cdot\frac{dy}{dt}.$$

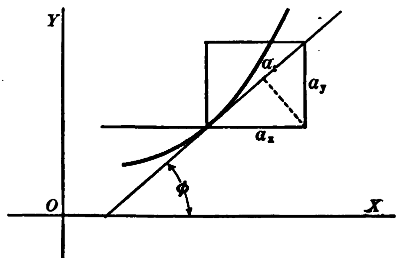


FIG. 100

This may be written

$$\frac{d^2s}{dt^2} = \frac{d^2x}{dt^2}\cdot\frac{dx}{ds} + \frac{d^2y}{dt^2}\cdot\frac{dy}{ds} = a_x\cos\phi + a_y\sin\phi;$$

$$\therefore \frac{d^2s}{dt^2} = a_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

= the acceleration along the tangent.

Similarly if a_n is the acceleration along the normal,

$$a_n = a_y \cdot \cos\phi - a_x \sin\phi = \frac{d^2y}{dt^2}\cdot\frac{dx}{ds} - \frac{d^2x}{dt^2}\cdot\frac{dy}{ds}. \quad . \quad . \quad (56)$$

This assumes that the positive sense of the normal acceleration is toward the center of curvature of the curve. If ρ = the radius of curvature of the curve, it is shown in the calculus that

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left(\frac{ds}{dx}\right)^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

$$\therefore \frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

$$= \frac{\left(\frac{dx}{ds}\frac{d^2y}{dt^2} - \frac{dy}{ds}\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{ds}\right)^3\left(\frac{ds}{dt}\right)^2} = \frac{a_n}{\left(\frac{dx}{ds}\right)^3\left(\frac{ds}{dt}\right)^2}.$$

Substituting this value of $\frac{d^2y}{dx^2}$, and $\frac{ds}{dt} = v$, in (a), we get

$$a_n = \frac{v^2}{\rho}. \qquad (57)$$

Let a denote the resultant acceleration, which we shall call the *total acceleration.* Then from Art. 164,

$$a = \sqrt{a^2_x + a^2_y}.$$

Similarly,

$$a = \sqrt{a^2_n + a^2_t}.$$

The student will note that if the particle is describing a straight line, the total acceleration, a, equals the acceleration along the path, for then $a_n = 0$. Further, that in all other cases the total acceleration is different from the tangential acceleration or the acceleration along the curve.

It will be further noted that according to Newton's Laws of Motion the total acceleration is directed toward the concave side of the curve. For, the "change of motion," according to the Second Law, is in the "direction of the acting force." So that if any force, whose line of action is not along the tangent to the path, acts on the particle, it will deviate from the tangent in that direction; that is, toward the concave side of the curve. Hence the normal acceleration is directed toward the center of curvature. The student will avoid difficulty if he uses equation (57) to obtain the magnitude of the normal acceleration and the fact just stated to obtain its direction.

To illustrate, let us find a_t, a_n, a_x, a_y, and a, for a particle P, describing a circle with uniform speed. Let the rectangular coördinates of P at any time, t, be x, y, and its polar coördinates be r, θ, r being the radius of the circle.

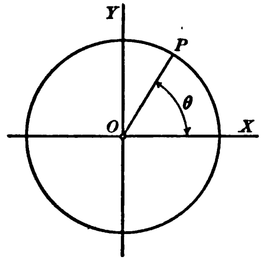


Fig. 101

Then,

$$\frac{ds}{dt} = r\frac{d\theta}{dt} = \text{a constant.}$$

$$\therefore \frac{d^2s}{dt^2} = 0 = r\frac{d^2\theta}{dt^2} = a_t.$$

$$x = r\cos\theta;\ y = r\sin\theta;$$

$$-\frac{dx}{dt} = r\sin\theta\,\frac{d\theta}{dt};\ \frac{dy}{dt} = r\cos\theta\,\frac{d\theta}{dt};$$

$$a_x = \frac{d^2x}{dt^2} = -r\cos\theta\left(\frac{d\theta}{dt}\right)^2,\ \text{since}\left(\frac{d^2\theta}{dt^2}\right) = 0.$$

$$a_y = \frac{d^2y}{dt^2} = -r\sin\theta\left(\frac{d\theta}{dt}\right)^2.$$

$$a = r\left(\frac{d\theta}{dt}\right)^2 = \frac{v^2}{r} = a_n.$$

The student will note that although P is moving with uniform speed, the acceleration is not zero.

168. Circular Motion, Angular Velocity, Angular Acceleration.— The motion of a particle in a circle may be treated more simply in a direct way as follows:

Let the particle move in a circle with a fixed center O and radius r. Let the position of the particle at any time be P. Let the arc $AP = s$, and let the angle $AOP = \theta$. Then the distance $s =$ the arc $AP = r\theta$, where θ is measured in radians. Differentiating the expression,

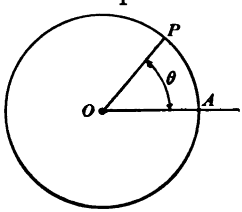


Fig. 102

$$s = r\theta,$$

we obtain the tangential velocity of the particle. That is,

$$v = \frac{ds}{dt} = r \cdot \frac{d\theta}{dt} = r\omega,$$

where

$$\omega = \frac{d\theta}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

The quantity $\frac{d\theta}{dt}$, designated by ω, is called the *angular velocity* of the particle, and is expressed in radians per second.

To obtain the tangential acceleration, we differentiate equation

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and obtain

$$a_t = \frac{dv}{dt} = r\frac{d^2\theta}{dt^2} = r \cdot \frac{d\omega}{dt} = r\alpha, \quad . \quad . \quad . \quad . \quad (b)$$

where

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

The quantity $\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$, designated by α, is called the angular acceleration of the particle, and is measured in radians per second per second.

Since

$$vdv = ads,$$

we can write

$$\omega d\omega = \alpha d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

We have two sets of analogous equations, one for linear and one for angular motion. They are:

For linear motion		For angular motion	
$v = \frac{ds}{dt}$.	(2)	$\omega = \frac{d\theta}{dt}$.	(58)
$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$. . .	(4)	$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$. . .	(59)
$vdv = ads$.	(34)	$\omega d\omega = \alpha d\theta$.	(60)

Equations (*a*) and (*b*) show the relation between these two sets.

NOTE. — The student will observe that if the particle is moving in any curve, and if the center of curvature of the curve at the point P is substituted for the fixed point O, the equations (58), (59), and (60) are true, the angle θ being measured from a normal to the curve at P, and the radius of curvature at the point being substituted for r.

If the particle be traveling in any other curve than a circle, and O is fixed, the quantities $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ are still called the angular velocity and the angular acceleration respectively; but the equations of this article are no longer true. It is easy to see that in most cases of moving machinery the various particles are describing circles about a fixed center.

169. Exercises

1. The velocity of a particle moving in a circular path whose radius is 3 feet changes from 7 revolutions to 4 revolutions per second in 5 seconds. If the change in speed is uniform: (*a*) What is the angular acceleration? (*b*) How many revolutions will it make before coming to rest? (*c*) What was the normal acceleration at the beginning?

2. The velocity of a flywheel is increasing from rest at the uniform rate of 2 radians per second. The radius of the wheel is 4 feet. How many seconds will elapse and how many radians be turned through before a point on the rim has a tangential velocity of 1200 feet per minute?

3. A grindstone with a radius of 18 inches is making 180 revolutions per minute. If it comes to rest with uniform retardation after making $7\frac{1}{2}$ complete revolutions, find the angular acceleration, the distance traveled by a point on the rim, and the time required for the grindstone to come to rest.

170. Acceleration along the Radius Vector; Perpendicular to the Radius Vector. — It is sometimes desirable to know the acceleration a_r parallel to the radius vector, and the acceleration a_θ perpendicular to the radius vector.

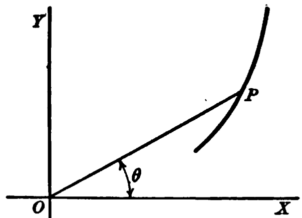


FIG. 103

Let P be any point on a curve and let x, y and r, θ be its rectangular and polar coördinates.

Then,

$$x = r\cos\theta,$$

$$y = r\sin\theta.$$

$$\therefore \frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}.$$

$$\left.\begin{aligned}&\therefore \frac{d^2x}{dt^2} = \frac{d^2r}{dt^2}\cos\theta - 2\sin\theta\frac{dr}{dt}\frac{d\theta}{dt} - r\sin\theta\frac{d^2\theta}{dt^2} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2;\\ &\text{similarly,}\\ &\frac{d^2y}{dt^2} = \frac{d^2r}{dt^2}\sin\theta + 2\cos\theta\frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\cos\theta\frac{d^2\theta}{dt^2} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2.\end{aligned}\right\}(a)$$

Now,

$$a_r = \frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta, \text{ and } a_\theta = \frac{d^2y}{dt^2}\cos\theta - \frac{d^2x}{dt^2}\sin\theta; \quad (b)$$

substituting the values of $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ as given in (a) into (b), we obtain

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2;$$

and

$$a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt}\frac{dr}{dt} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).$$

171. The suggestions of Art. 144 for the solution of problems in rectilinear motion apply with equal force to the solution of problems in curvilinear motion. However, in the former case, we wrote one equation of motion, while in the latter it is, in general, easier to write equations of motion along two lines at right angles to each other. In general, if the path of the particle is known, resolve along the tangent and the normal; if the path is unknown, resolve along the x- and y-axes. In any case *insert every force* acting on the particle, and resolve each force along the lines chosen.

By Newton's Second Law of Motion

the *mass* of a particle $\times$ its acceleration *along any line* = the resolved parts of the forces *acting on it along that line.*

If the forces have been resolved along the tangent and the normal, let the sum of the resolved parts along the tangent equal ΣT_i, and those along the normal $= \Sigma N_i$. Then

$$\Sigma T_i = \frac{W}{g} \text{ times the tangential acceleration}$$
$$= \frac{W}{g} \cdot a_t.$$
$$\Sigma N_i = \frac{W}{g} \text{ times the normal acceleration} = \frac{W}{g} \cdot \frac{v^2}{\rho}.$$

This gives two equations of motion, and if both involve the acceleration, four constants of integration will be introduced and hence four initial conditions must be given.

We will add the *résumé:*

1. Write the equations of motion.
2. Integrate them.
3. Determine the constants of integration.

172. Motion on an Inclined Plane. — To illustrate, consider the motion of a particle on an inclined plane. Suppose a body of weight W lies on a smooth plane making an angle α with the horizon. Let s be measured along the plane from O positive upward. Then the force acting on the body parallel to the plane is $W \sin \alpha$.

Hence, the equation of motion is,

$$\frac{W}{g} a = - W \sin \alpha.$$
$$a = - g \sin \alpha.$$

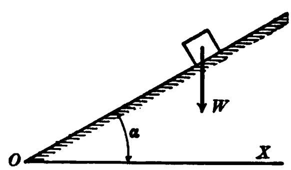


FIG. 104

Since $g \sin \alpha$ is a constant, this equation is of the form integrated in Art. 147. The student may find the relation between the space and the velocity, the space and the time, and

the velocity and the time, in the cases 1, 2, and 3 below, assuming that the particle starts from a point s_0 on positive side of O.

1. Assume the body starts from rest.
2. Assume the body is projected *up* the plane with a velocity v_0.
3. Assume the body is projected *down* the plane with velocity v_0.
4. How far will the body in 2 ascend the plane?
5. Discuss completely the motion of the particle if the plane is rough, using the assumption of Art. 145 (6).

173. Exercises

1. A block is projected down a plane inclined 45° to the horizon, with an initial velocity of 22 feet per second. Find the velocity at the end of 7 seconds, and the distance traveled by the block during the same time.

2. From a point on a plane inclined 30° to the horizon, a body is made to slip up the plane with a velocity of 16.1 ft. per second. How far will it go before it comes to rest? How far will it be from the starting point, 5 seconds after the beginning of motion? (Assume $g = 32.2$.)

Ans. (1) 8.05 ft. (2) 120.75 ft. below the starting point.

3. Two blocks, A and B, weighing 10 pounds and 16 pounds respectively are sliding down a plane inclined 40° to the horizon. There is a constant friction of two pounds between block A and the plane, but no friction between block B and the plane. If the blocks are at rest, and just in contact at the beginning, B being above A, how far will the blocks have moved at the end of 4 seconds? What will be the pressure between the blocks?

4. A block weighing 32.2 pounds slides down a double inclined plane. The first plane, which is 40 feet long, makes an angle of 45° with the horizon; and the second, 30 feet long,

makes an angle of 30° with the horizon. The friction between the block and the planes is respectively 6 pounds and 3 pounds. If there is no shock in changing direction, how long would the block be in traversing the two planes? With what velocity would it reach the bottom of the second plane?

174. Motion of a Projectile. — The theory of plane motion is illustrated admirably by the following problem:

A particle is projected from a point O with an initial velocity v_0 along a line OT, making an angle α with the horizontal. Suppose the particle moves *in vacuo* and therefore is subject to no resistance of the air. Find its path. Let us take the vertical plane through OT as the xy-plane, and the x-axis horizontal.

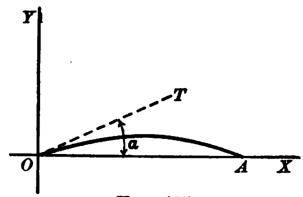


FIG. 105

The particle, *when in motion*, is acted on by only one force, viz. its weight, which is parallel to the y-axis. Hence the equations of motion are

$$\frac{d^2x}{dt^2}=0, \qquad (a)$$

$$\frac{W}{g}a_y=\frac{W}{g}\frac{d^2y}{dt^2}=-W.$$

$$\therefore \frac{d^2y}{dt^2}=-g. \qquad (b)$$

These equations are true simultaneously, and must be integrated as simultaneous equations.

Integrating (a) and (b),

$$\frac{dx}{dt}=c_1,$$

$$\frac{dy}{dt}=-gt+c_2,$$

where c_1 and c_2 are constants of integration. We may determine c_1 and c_2 by noting that when $t = 0$ the velocity along the x-axis and the y-axis is respectively $v_0 \cos \alpha$ and $v_0 \sin \alpha$.

$$\therefore c_1 = v_0 \cos \alpha,$$
$$c_2 = v_0 \sin \alpha.$$

$$\therefore \frac{dx}{dt} = v_0 \cos \alpha, \qquad (c)$$

$$\frac{dy}{dt} = -gt + v_0 \sin \alpha. \qquad (d)$$

Integrating (c) and (d),

$$x = v_0 \cos \alpha t + c_3,$$

$$y = -\frac{gt^2}{2} + v_0 \sin \alpha t + c_4,$$

where c_3 and c_4 are constants of integration. To determine them, remember that when $t = 0$, $x = y = 0$.

$$\therefore c_3 = c_4 = 0.$$

Hence,
$$x = v_0 \cos \alpha t, \qquad (e)$$

$$y = \frac{-gt^2}{2} + v_0 \sin \alpha t. \qquad (f)$$

Eliminating t from (e) and (f), we get

$$y = x \tan \alpha - \frac{gx^2}{2 v_0^2 \cos^2 \alpha}. \qquad (g)$$

Hence, the path described by the particle is a parabola, whose vertex is given by the equations

$$x_v = \frac{v_0^2}{g} \sin \alpha \cos \alpha,$$

$$y_v = \frac{v_0^2 \sin^2 \alpha}{2g}.$$

To find the place where the particle will strike a horizontal plane through the origin, put $y = 0$ in (g) and solve for x. The result is, if we put this value of $x = R$,

$$R = \frac{v_0^2 \sin 2\alpha}{g}. \qquad (h)$$

The distance OA (see Fig. 104) $= R$ is called the *range* of the particle on the plane OX.

To find the *time of flight*, that is, the time it takes the particle to reach A, put $y = 0$ in (f), and solve for t.

If $\qquad t_1 =$ time of flight,

$$t_1 = \frac{2\,v_0}{g}\sin\alpha. \qquad (i)$$

To find the angle α, such that the range be greatest, differentiate R in equation (h) with respect to α, and put

$$\frac{dR}{d\alpha} = 0. \quad \text{That is,}$$

$$\frac{dR}{d\alpha} = \frac{v_0^2}{g}2\cos 2\,\alpha = 0.$$

$$\therefore \cos 2\,\alpha = 0,$$

$$2\,\alpha = 90°.$$

$$\therefore \alpha = 45°.$$

175. Another problem of this type is to find the path of a particle that is attracted to a fixed center O by a force that varies inversely as the square of the distance of the particle from the center.

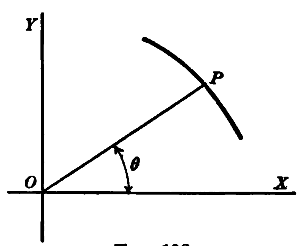


FIG. 106

Take the fixed center O as origin, and let P, whose coördinates are x, y be the position of the particle. Let $OP = r$. Let $\theta =$ the angle which OP makes with the x-axis; then the equations of motion are

$$\frac{W}{g}\frac{d^2x}{dt^2} = -\frac{k^2}{r^2}\cos\theta = -\frac{k^2}{r^2}\cdot\frac{x}{r} = -\frac{k^2x}{r^3}, \qquad (a)$$

$$\frac{W}{g}\frac{d^2y}{dt^2} = -\frac{k^2y}{r^3}, \qquad (b)$$

where $\qquad r^2 = x^2 + y^2.$

The student will note that equations (*a*) and (*b*) each contain three variables, x, y, and t; r being replaced by $\sqrt{x^2 + y^2}$. These equations, therefore, cannot be integrated separately, as in the preceding article. They can be integrated, however, by methods that for lack of space we shall not explain. Suffice it to say that when integrated the path turns out to be a conic section with O as one focus. This problem is at the basis of all celestial mechanics treating of the motions of two mutually attracting bodies around each other, such as the sun and a planet, the sun and a comet, a planet and a satellite, double stars, etc.

The equations are integrated in Art. 54, Moulton's *Celestial Mechanics*, and in many other places. A force that always acts through a fixed point is called a *central force.*

176. Exercises

1. A ball fired from a gun which was inclined 40° to the horizon, struck the ground at a distance of 2500 feet from the gun. Find (1) the velocity with which it left the gun; (2) the time of flight.

2. A ball is fired from a gun set so as to have the greatest range. It leaves the gun with a velocity of 1200 feet per second. Find (1) the range, (2) the time of flight.

Ans. (1) 8.47 miles (nearly); (2) 52.7 seconds (nearly).

3. A ball fired with a velocity v_0 at an inclination of α to the horizon just clears a vertical wall which subtends an angle β at the point of projection. Find the distance between the foot of the wall and the point where the ball strikes the ground.

Ans. $\dfrac{2\,v_0^2}{g}\cos^2\alpha\tan\beta.$

4. A cannon set at the foot of a hill is fired up the hill at a target set on the slope of the hill. Given the velocity v_0 with which the ball left the gun as 1500 feet per second, the inclination to the horizon of a line joining the target and the

gun as 15°, and the distance of the target from the gun measured along this line as 3 miles. Find the angle of elevation of the gun, that the ball may hit the target, neglecting the resistance of the air.

5. From a fort 2000 feet above sea level a cannon is fired at a target floating on the sea. The horizontal distance between the cannon and the target is 5 miles. If the angle of elevation of the gun is 50°, find the velocity with which the ball left the cannon in order to strike the target.

6. A train is moving horizontally at the rate of 30 mi. per hour. A pebble is dropped from the window. If the ground be 10 ft. below the window, at what angle will it strike the ground? *Ans.* At an angle of 29° 58′ with the horizon.

7. A body of snow is lying on a roof 20 feet from the eaves, which are 20 feet from the ground. If the snow, starting from rest, slides without friction down the roof, where will it strike the ground, the rafters making an angle of 30° with the horizon? *Ans.* $10\sqrt{3}$ feet.

CHAPTER XI

WORK AND ENERGY

177. Work. — If a force F, *constant in magnitude* and *direction,* acts on a particle at P, and if the particle is displaced to a point P' such that $PP' = s$ and makes an angle θ with the line of action of F, then the work w, done by the force F, is defined by

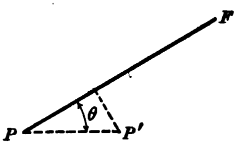


FIG. 107

$$w = F \cdot s \cos \theta. \quad . \quad . \qquad (61)$$

That is, the work equals the *product of the force and the projection of the displacement of the point of application of the force on its line of action.*

Equation (61) may be written

$$w = F \cos \theta \cdot s. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (62)$$

That is, the work done by a force equals the *product of the resolved part of the force along the line of the displacement multiplied by the displacement.*

Work is said to be positive if the projection of PP' on the line of action of the force is in the direction in which the force acts, and negative if the projection of PP' on the line of action of the force is in the opposite direction. For example, if a particle whose weight is W falls through a height h, the work done by the force W is $w = + Wh$. If, on the other hand, a particle whose weight is W ascends through a height h, the work done by W is $w = - Wh$.

If a particle whose weight is W moves a distance PP' down a plane inclined to the horizon by an angle α, the work done by the weight W is $w = W \cdot PP' \sin \alpha$. If on the other

hand it moves up the plane to P'', the work done by W is

$$w = - W \cdot PP'' \sin \alpha.$$

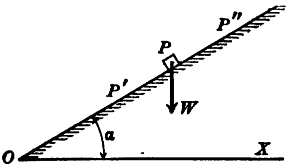


FIG. 108

It is evident from (62) that $w = 0$: (1) if the displacement is zero, *i.e.* the point of application does not move; or (2) if F is zero; or (3) if the displacement is *perpendicular to the line of action* of the force, and therefore $\cos \theta = 0$.

Moreover, if the displacement is along the line of action of the force, $\cos \theta = 1$, and

$$w = F \cdot s. \quad \ldots \ldots \ldots \quad (63)$$

In this very important case,

$$\text{work} = \text{force} \times \text{space}, \quad \ldots \ldots \quad (63)$$

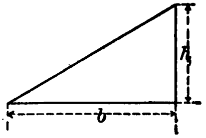
where space is used as the length of the path described by the particle as defined in Art. 7.

178. **Exercises**

1. A body weighing 6 tons falls vertically through a distance of 10 feet. Find the work done by its weight.

2. A cylindrical well 100 feet deep, with a radius of 2 feet, is half full of water. How much work will it require to raise the water to the top? Water weighs 62.5 pounds per cubic foot.

3. How much work will it require to raise the materials for a retaining wall 60 feet long, 20 feet high, 7 feet thick, the materials weighing 150 pounds per cubic foot?



4. A body weighing W pounds is pulled up a smooth inclined plane with dimensions as marked in the figure. Show that the work done is the same as would be required to raise the weight through the *vertical distance h.*

5. Assuming that the plane of exercise 4 is rough, show that the work of moving the body up the inclined plane is the same as moving it over an equally rough horizontal plane of length b, and raising it through a vertical height h.

179. Work of More than One Force. — To find the work done by any number of forces (constant in magnitude and direction) acting on a particle which is displaced through a distance PP', we must multiply the sum of the resolved parts of the forces along PP', by PP'. But, by the polygon of forces (Art. 22) the sum of the resolved parts, along any line, of any number of forces acting on the particle equals the resolved part of their resultant along that line. Hence the

THEOREM: *The sum of the work done by any number of forces acting on a particle equals the work done by their resultant.*

In what follows, therefore, we lose nothing in generality if we consider a single force only.

180. *The unit of work* is the work done by a unit force acting while a particle moves through a unit distance parallel to the force. American engineers usually employ as the unit of work the *foot-pound*, that is, the work done by a force of one pound acting through the distance of one foot. The c. g. s. unit of work (Art. 143) is the work done by one dyne acting while the body moves one centimeter parallel to the force. It is called the *erg*. Another unit of work, called the *joule*, is defined by

$$\text{one joule} = 10^7 \text{ ergs}.$$

The conversion factor (Art. 140) evidently is $[F]\,[l] = \left[\frac{ml^2}{t^2}\right]$, which enables one to convert easily quantities expressed in the units of one system into expressions involving the units of any other system. The unit of work which we shall use is the foot-pound.

181. Work done by Forces Constant in Direction but Variable in Magnitude. — We have considered the work done by forces constant in direction and magnitude, both when the displacement coincides with the line of action of the resultant force (equation 63), and when the displacement makes an angle θ with the resultant force (equation 62). In this article, we shall take up the work done by forces variable in magnitude but constant in direction; and in a later article, the work done by forces which vary both in direction and magnitude.

Let us consider the case of a particle P acted upon by a variable force F, the displacement being along the x-axis.

FIG. 109

Let dw be the work done by such a force while the particle is being displaced a distance dx. We shall assume that dx may be chosen so small that F is constant while the particle moves through that distance. Then

$$dw = Fdx.$$

The work done in displacing the body from x_0 to x_1 would therefore equal

$$w = \int_0^{x_1} Fdx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

To evaluate the integral $\int_{x_0}^{x} Fdx$, both graphical and analytic methods may be employed. If the force F can be expressed as a function of x, the displacement, the analytic method will usually be found simpler, since the value of F can be substituted and the integral immediately evaluated.

To illustrate this, let us consider the work necessary to stretch a horizontal steel bar, originally under no constraint, a certain distance e, within the elastic limit of the material. Since the bar obeys Hooke's Law (Art. 153), the force F required to stretch the bar would equal

$$F = kx,$$

where x = amount the bar is stretched, and

k = a constant depending upon the physical properties of the bar.

If we let $w =$ total amount of work done,

$$w = \int_0^e kx dx = \frac{ke^2}{2}.$$

Let us assume that the bar had already been stretched an amount, e_0, and that it is required to find the work done to stretch it to e_1, within the limits of Hooke's Law.

Then
$$w = \int_{e_0}^{e_1} kx dx = \frac{k(e_1^2 - e_0^2)}{2}.$$

These results could have been obtained graphically with little if any more labor. From the form of the integral $\int_{x_0}^{x_1} F dx$, it is evident that to represent work by an area it is only necessary to plot a curve in which the abscissæ represent the displacement, and the ordinates the value of F. The work will then equal the area between the curve and the x-axis. In the case of the bar, the equation $F = kx$ being of the first degree in x, the curve will be a straight line and will pass through the origin. The final value of F is ke. Therefore the area (Fig. 110), and hence the work, is $w = \frac{ke^2}{2}$.

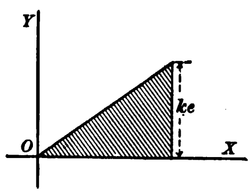


FIG. 110

The work required to stretch the bar from e_0 to e_1 is represented by the shaded area in Fig. 111. It is evident from the figure that the average value of $F = \frac{ke_1 + ke_0}{2}$. This is the mean of the two bases of the trapezoid in the figure. The area and consequently the work

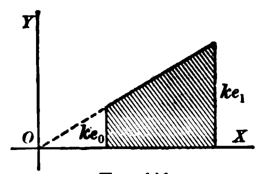


FIG 111

$$w = \frac{(ke_1 + ke_0)}{2}(e_1 - e_0) = \frac{k}{2}(e_1^2 - e_0^2).$$

As a second illustration of a variable force acting upon a body moving in a straight line, let us consider the pressure of the steam on the piston head of a steam engine. The volume occupied by the steam depends only on the distance x that the piston head has traveled, and since after the cut-off the pressure is a function of the volume, the pressure at any instant is a function of x, and therefore a variable.

The work done by the steam pressure may therefore be represented graphically by taking as ordinates of a curve the pressure of the steam on the piston, and as the corresponding abscissæ the distances the piston head has traveled (see Figs. 112 *a*, 112 *b*). Then the work on the forward stroke of the piston equals the area between the curve and the x-axis. If, on the backward stroke, the piston head moves against a pressure, this pressure will do negative work, which is also represented by the area between a curve and the x-axis. The difference of the two areas is the work of the stroke.

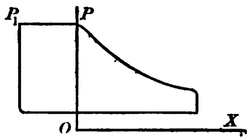


FIG. 112 *a*

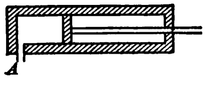


FIG. 112 *b*

In actual practice, the pressure is recorded by a pencil moved vertically by a guide controlled by the pressure of the steam in the piston box. The pencil marks on a paper which is wrapped around a vertical cylinder that revolves around its axis in one direction during the forward stroke of the piston, and in the opposite direction during the backward stroke. While the steam is entering the cylinder at A (Fig. 112 *b*), the pressure being approximately constant and equal to the boiler pressure, the pencil then describes the curve P_1P. When the port at A is closed, the pressure which is due only to the expansion of the steam, varies as x, the distance the piston head has traveled after the steam is shut off, so that when the flywheel has made one complete revolution, the pencil, under ideal conditions, will have described a closed curve, resembling approxi-

mately Fig. 112 *a*. To find the work done on the piston head we need only measure the area of this curve.

182. Exercises

1. If it requires a force of 40 pounds to compress a spring one inch, how many foot-pounds of work are required to compress it 15 inches? What will be the final pressure on the spring?

2. A particle is being pulled along a rough horizontal plane by a force acting parallel to the plane. The roughness is such that the fricton $= 4x^2$, x being the distance in feet from the initial point. How much work will the force do in traveling 16 feet?

3. Find the work required to draw a body weighing 500 pounds 100 feet up a plane making an angle of 30° with the horizon, if the friction between the body and the plane equals $(s^2 + 3s + 2)$ pounds, where s equals the distance traveled.

4. A steam engine has a cylinder 18 inches long with a radius of 6 inches. Steam with a boiler pressure of 120 pounds per square inch is admitted into the cylinder during the first third of the stroke. The port is then closed, and assuming that the pressure due to the expansion of the steam obeys Boyle's Law, which is the *pressure times the volume equals a constant*, find the work done on the piston during the forward stroke.

183. Work of a Force varying in Direction and Magnitude. Plane Motion. — Let a particle move in a plane curve subject to the action of a force F.

Let X, Y be the resolved part of F along the x- and y-axes respectively.

Let $\phi =$ the angle the tangent to the curve at any point x, y makes with the x-axis. Let ds be the displacement of the

particle along the curve. Then the work w, done by the forces X and Y, is (Art. 177)

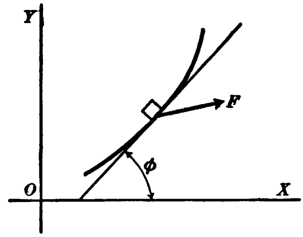


FIG. 113

$$w = \int (X \cos \phi + Y \sin \phi)\, ds$$
$$= \int \left(X \frac{dx}{ds} + Y \frac{dy}{ds} \right) ds$$
$$= \int (Xdx + Ydy).$$

A discussion of the expression $\int (Xdx + Ydy)$ is found in Art. 187.

184. Power is the Rate of doing Work.— It is often necessary in problems of engineering to specify the time in which a certain amount of work is to be done. The unit most commonly used in practice is the *Horse-power*, defined by Watt, and designated by H.P. One horse-power = 33,000 foot-pounds per minute, or 550 foot-pounds per second, or 396,000 inch-pounds per minute.

Thus, to raise 100 tons, vertically, through 66 feet requires $100 \times 2000 \times 66$ foot-pounds of work. To do this work in 5 minutes requires

$$\frac{100 \times 2000 \times 66}{33000 \times 5} = 80 \text{ H.P.}$$

185. Exercises

1. What power would be required to empty the well of exercise 2, Art. 178, in one hour, neglecting all losses due to friction?

2. A steamship of 22,000 horse-power runs 3300 miles in six days. Find the resistance to the ship's motion, assuming the resistance is constant. *Ans.* Resistance = 180 tons.

3. If a boy weighing 130 pounds can climb to the top of a water tower 129 feet in height in four minutes, what horse-power is he developing?

186. The Theorem of Kinetic Energy and Work.

DEFINITION.—The kinetic energy of a particle equals $\frac{W}{2g}v^2$.

THEOREM. — *The change in the kinetic energy of a particle equals the work done by the resultant force acting on it.*

We shall consider two cases:

(*a*) When the particle is moving in a straight line.

(*b*) When the particle is moving in a plane curve.

(*a*) A particle whose weight is W is moving along a straight line. Then the resultant F of all the forces acting on the particle is directed along the line of motion (Art. 135). If v is the velocity and a the acceleration of the particle,

$$\frac{W}{g}vdv = \frac{W}{g}ads. \quad \text{(Art. 138.)}$$

But

$$F = \frac{W}{g}a. \quad \text{(Art. 142.)}$$

$$\therefore \frac{Wv^2}{2g} = \int Fds + c.$$

If $v = v_0$ when $s = s_0$, we may eliminate c and obtain

$$\frac{W}{2g}(v^2 - v_0^2) = \int_{s_0}^{s} Fds. \quad \text{. . . . . (65)}$$

By definition $\frac{W}{2g}v^2$ is called the *kinetic energy* of the particle and $\int_{s_0}^{s} Fds$ equals the work done by the resultant force F, while the particle moves from the position s_0 to the position s, which was to be proved. It may be noticed that this equation is independent of the time. Moreover, if F is constant, equation (65) becomes

$$\frac{W}{2g}(v^2 - v_0^2) = F(s - s_0). \quad \text{. . . . . (66)}$$

(*b*) We shall now consider the motion of a particle in a plane curve. Let it be acted upon by any number of forces,

and let the resultant force, including the reaction of the curve, be F. Let the resolved part of F along the x- and y-axes be X and Y respectively. At any time t let x and y be the coördinates of the particle, and v, v_x, v_y be the velocity of the particle along the curve and along the x- and y-axes respectively. Let a_x and a_y be the acceleration of the particle along the x- and y-axes respectively.

Then
$$v_x dv_x = a_x dx.$$
$$v_y dv_y = a_y dy.$$

But
$$X = \frac{W}{g} a_x;$$
$$Y = \frac{W}{g} a_y;$$
$$\therefore \frac{W}{2g}(v_x^2 + v_y^2) = \frac{W}{2g} v^2 = \int (Xdx + Ydy) + c.$$

To determine c, let $v = v_0$ when $x = x_0$ and $y = y_0$.

Hence, $\dfrac{W}{2g}(v^2 - v_0^2) = \displaystyle\int_{x_0 y_0}^{xy} (Xdx + Ydy),$

which proves the proposition for motion in a plane curve.

NOTE. — The proof given here is for plane motion. An analogous proof might be given for motion in space.

187. The Expression $\int (Xdx + Ydy)$.

(1) If the force acting on the particle is some function of its coördinates and the equation of the path is known, the expression $\int (Xdx + Ydy)$ can be transformed into one involving one variable only, and in many instances the integral can be evaluated; for, from the equation of the path one of the variables, y, can generally be expressed in terms of the other, x, and having substituted this value of y in X and Y, we obtain a quantity under the integral sign which involves one variable

Hence we may state the following

THEOREM. — *If* X *and* Y *are functions of the coördinates of the position of the particle only* (*that is, do not depend on the direction or the magnitude of the velocity, nor the time*), *and if* $\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$, *the system of forces is conservative, and a function* U *exists such that*

$$X = \frac{\partial U}{\partial x}, \text{ and } Y = \frac{\partial U}{\partial y}.$$

For example, consider a particle moving *in vacuo* in a vertical line under the action of gravity, supposed constant. Suppose the y-axis is a vertical line positive upward; then

$$X = 0; \quad Y = -W \text{ (the weight of the particle)};$$

$\therefore$ equation (68) is satisfied.

$$U = -Wy;$$

and

$$\frac{W}{2g}(v^2 - v_0^2) = -W(y - y_0).$$

The student may show that in the problem of the elastic string (Art. 155) the system of forces is conservative. It is a noteworthy fact, that $\int(Xdx + Ydy)$ is in very many cases an exact integral.

188. It follows also that if a particle, subject to the action of forces, for which a force function exists, moves from a position A to a position B, the change of kinetic energy depends only on the initial and final position of the particle and not at all on the path by which the transition is made. That is, Fig. 115, the change in kinetic energy is the same whether the particle pass from A to B over the path L, the path M, or the path N. Moreover,

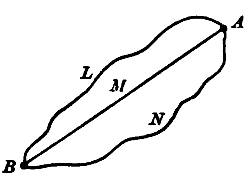


FIG. 115

if the body in its motion return to its initial position, it will have the same kinetic energy as it did in the beginning, and, indeed, should it pass through any point more than once, it will have the same velocity on each passage through that point whatever way it may be going.

189. It is usual to call the negative of the function U the *potential energy* of the particle. If we denote the potential energy by V and the kinetic energy by T, we shall have for a conservative system at any time,

$$T + V = T_0 + V_0. \qquad (69)$$

This is the law of *Conservation of Energy* for a conservative system of forces.

It may be translated in various ways. For example, if a particle moves subject to the action of a conservative system of forces, the sum of the kinetic energy and the potential energy is constant throughout the motion. Or, the amount of kinetic energy gained (or lost) equals the amount of potential energy lost (or gained).

Another point worthy of notice is that the kinetic energy of a particle is expressed in terms of its velocity, and the potential energy in terms of its coördinates.

190. Examples of non-conservative systems of forces are furnished by the motion of a particle on a rough surface, where the friction depends not on the position, but on the direction, and to a certain extent on the magnitude of the velocity of the particle; or, by the motion of bodies through a resisting medium such as a particle in the air, a railway train, a vessel moving through the water, etc.

191. The Energy of a Particle is its Ability to do Work. — We have shown that the change of the kinetic energy of a particle equals the work done by the forces acting on it.

It seems reasonable from the Third Law of Motion that the equations

$$\frac{W}{2g}(v^2 - v_0^2) = \int F ds = w,$$

and
$$\frac{W}{2g}(v^2 - v_0^2) = \int (X dx + Y dy) = w,$$

may be read the other way, viz. that in changing from the velocity v to the velocity v_0, the particle will do work on another body equal to w. This comes to saying that a particle will do as much work in giving up its velocity as has been done on the particle in order that it might acquire it. This, roughly, is the great axiom known as the Conservation of Energy, usually assumed by physicists, and, stated broadly, is that energy is indestructible. A stone falling from a great height acquires a velocity which becomes zero on striking a floor. This energy $\frac{W}{2g}v^2$, which the stone acquired, has been used in producing sound, heat, breaking the stone and the floor. The energy which a volume of water acquires in falling is partly communicated to a water wheel, partially used in other ways. No machine is able to turn into useful work, owing to friction and other causes, all the energy communicated to it. The ratio of the "useful" work done by a machine to total energy communicated to it is called the efficiency of the machine.

192. Working Forces and Resistances. — Let us consider a particle acted upon by any number of forces. Let the particle move in a straight line through a distance s. From the definition of positive and negative work, it is evident that if the projection of the displacement upon the line of action of the force is in the direction of the force, that particular force is helping to move the particle and can be called a "working force." On the contrary, if the projection falls in the opposite direction, that force is doing negative work and hence is resisting the motion and can be called a "resistance."

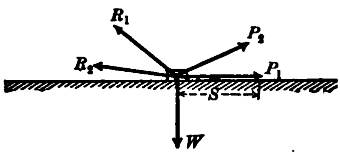


FIG. 116

193. To illustrate, consider the problem: A weight W is moving along a rough horizontal plane acted upon by the forces, P_1, P_2, R_1, and R_2, making angles with the plane of α_1, α_2, ϕ_1, and ϕ_2 respectively. Let s be the displacement, v_0 and v_1 the initial and final velocities respectively. Then

$$P_1 s \cos \alpha_1 + P_2 s \cos \alpha_2 = R_1 s \cos \phi_1 + R_2 s \cos \phi_2 + \frac{W}{2g} v_1^2 - \frac{W}{2g} v_0^2.$$

That is, *the work done by the working forces = work done by the resistances + the change in kinetic energy.*

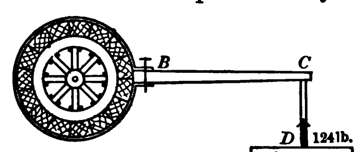
194. **Exercises**

1. Find the work done by a locomotive that changes the velocity of a train weighing 150 tons from 10 miles per hour to 40 miles per hour in 1 minute, the frictional resistance being 10 pounds per ton.

2. Show that the forces acting on the particle in Art. 175 are conservative, and find one integral of the equation of motion.

3. Find the force function in case of a particle moving as in Art. 174.

4. *The Prony Brake.* — The flywheel of an engine makes 100 revolutions per minute against the friction produced by a prony brake. If the radius of the flywheel is 2 feet and the distance from the center of the wheel to the scales 5 feet, and the scales record 124 pounds pressure, neglecting the weight of the brake itself, what H.P. is the engine developing?



NOTE. — The prony brake shown in the sketch consists of blocks of wood held against the rim of the flywheel by means of a piece of strap iron, one end of which is rigidly connected to the arm *BC*, and the other is held by means of a screw passing through it and the rigid arm. By tightening this screw, the tension in the strap iron and consequently the friction between the blocks of wood and the flywheel can be increased at will.

The friction between the blocks and the wheel tends to cause the entire brake to rotate with the flywheel. This tendency is resisted by the arm *CD*, which rests on the scales at *D*, the scales giving a measure in pounds of the resistance necessary. It will be noted that the lever arm of the friction between the brake and the flywheel is equal to its radius.

5. A steam engine has a cylinder 2 feet long with a radius of 5 inches. Steam with a boiler pressure of 100 pounds per square inch is admitted into the cylinder during the first half of the stroke. The valve is then closed and the pressure due to the expansion of the steam obeys the law p. v. = a constant.* Find the work done on the piston during the forward stroke. *Ans.* 13,310 foot-pounds (nearly).

6. If the flywheel is making 250 revolutions per minute, and the steam is admitted to the cylinder of exercise 5 from one end only, find the horse-power of the engine.

Ans. 101 H. P. (nearly).

7. The efficiency of a certain hydraulic motor is 80 per cent. If a lake supplies 60 cu. ft. of water per second to the wheel, the water having a fall of 15 feet, what power would the motor be able to deliver? *Ans.* 81.8 H. P.

8. Engineers for the government and for private corporations are investigating every stream in the United States to discover possibilities for the development of power. If they find that a certain stream has a cross section of 500 sq. ft. at a point where the mean velocity is 2 miles per hour and that at that point they can secure a fall of 35 feet, what power could be developed, neglecting all losses by friction, etc.?

* See exercise 4, Art. 182.

9. A railroad train made up of 10 cars is moving up a slope which rises 1 foot every 100 feet. If each car weighs 40 tons and there is a resistance due to friction and other causes of 12 pounds per ton, what work is done by the pull in the draw-bar of the locomotive while the train goes 1 mile up this grade? If the velocity of the train is constant at 20 miles per hour, what power does the draw-bar pull develop?

10. In problem 9, how much could the constant draw-bar pull be *diminished* if it were permissible to allow the velocity to decrease from 20 miles per hour at the foot of the slope to 10 miles per hour at the top of the slope? What power would be developed at 10 miles per hour?

CHAPTER XII

CONSTRAINED MOTION

195. A particle acted upon by any number of forces may, because of the reaction of a given curve or surface, be compelled to move along that curve or surface. This is known as a problem of *constrained motion.* The motion of a particle on an inclined plane considered in Art. 172 is a special case of this general problem. Many other illustrations readily suggest themselves, such as a train moving on a curved track, the bob of a pendulum, a piece of machinery moving in a groove, etc. We shall content ourselves with the consideration of motion on a given curve, considering plane curves only.

Motion on a given curve may be effected in a number of ways. The particle may be thought of as moving in a hollow tube, the diameter of the tube being just large enough to contain the particle; or, as if the particle were a ring moving on a wire in the form of a curve. However, in most problems the constraint is one-sided; that is, the particle will, *under the acting forces,* move on the concave side of the curve, and would, under the action of the given forces, leave the curve if it (the particle) were placed on the convex side of the curve, or conversely.

It is customary to treat the problem of constrained motion as follows. Since the curve changes or tends to change the motion of the particle, there is a force of reaction R of the curve on the particle which in general is different in magnitude and direction for every point of the curve. If we substitute this force in the place of the curve, the particle will describe the same path. We say then that "we have made the particle a free body." The problem may now be treated by the methods

of Chapter X. Let us consider the most general case of motion in a plane curve. A particle P, of weight W, is constrained to move in a plane curve C under the action of any number of forces, whose resultant F acts in the plane of the curve. F includes all forces except the reaction of the curve.

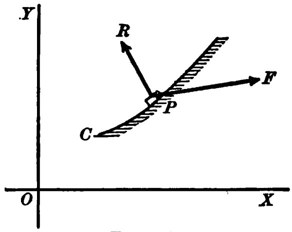


FIG. 117

Let the xy-plane be the plane of the curve. Let X_1 and Y_1 be the resolved part of F along the x- and y-axes, respectively.

Let R be the reaction of the curve, making an angle α with the positive direction of the x-axis.

The equations of motion of P are:

$$\frac{W}{g}a_x = \frac{W}{g}\frac{d^2x}{dt^2} = X_1 + R\cos\alpha;$$

$$\frac{W}{g}a_y = \frac{W}{g}\frac{d^2y}{dt^2} = Y_1 + R\sin\alpha. \tag{70}$$

If the curve is smooth, R will be directed along the radius of curvature.

196. The Pendulum. — One of the best illustrations of constrained motion is that of the pendulum.

One end of an inextensible string or rod without weight is fastened to a fixed point C, and from the other end a particle of weight W is suspended. If the particle is pulled aside, so that the string or rod is no longer vertical, and then let go, it will move in the arc of a circle, the center of which is C, and the radius, l, where l is the length of the string. This mechanism is called a *simple pendulum*.

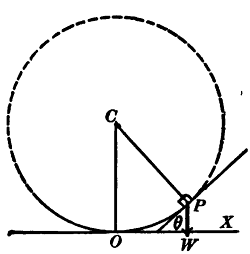


FIG. 118

To find the motion of the pendulum, take O, the lowest point in the circle as origin, the tangent at the point O as the x-axis, positive to the right. Take OC as

the y-axis, positive upward. Let T be the tension of the string; let θ, positive counter-clockwise, be the angle that the string makes with the vertical at any time.

Resolve the forces, acting on the particle, along the tangent. Since the tension of the string is normal to the circle, the resolved part of the tension along the tangent is zero. The resolved part of the weight along the tangent is $-W\sin\theta$. Hence, the equation of motion is

$$\frac{W}{g}\frac{d^2s}{dt^2} = -W\sin\theta,$$

or

$$\frac{d^2s}{dt^2} = -g\sin\theta.$$

Since

$$v = l\frac{d\theta}{dt},$$

$$\frac{dv}{dt} = \frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}.$$

Therefore the equation of motion becomes

$$\frac{d^2\theta}{dt^2} = \frac{-g}{l}\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

This equation cannot be integrated completely in finite terms. If, however, θ is so small (and in practice this is generally the case) that θ may be substituted for $\sin\theta$, this equation becomes

$$\frac{d^2\theta}{dt^2} = \frac{-g}{l}\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

which is the form integrated in Art. 151.

Hence, if

$$v = 0, \text{ when } \theta = \theta_0 \text{ and } t = 0,$$

$$\theta = \theta_0 \cos\sqrt{\frac{g}{l}}\,t.$$

Since

$$\frac{d^2\theta}{dt^2} = \frac{-g}{l}\theta = \alpha, \text{ and } \omega d\omega = \alpha d\theta \text{ (Art. 168)},$$

we have

$$\omega d\omega = \frac{-g}{l}\theta d\theta.$$

Integrating, we get

$$\omega = \frac{d\theta}{dt} = \pm\sqrt{\frac{g}{l}}\sqrt{\theta_0^2 - \theta^2}. \quad . \quad . \quad . \quad . \quad (73)$$

Equation (73) shows that the velocity $= 0$ when and only when $\theta = \pm\,\theta_0$, and that the smaller θ is, the greater the velocity is, and that the velocity is greatest when $\theta = 0$. Hence the particle vibrates, coming to rest at the same angular distance to the left and to the right of OC.

To find the time of a complete oscillation, that is, the time it requires for the particle to swing from the extreme right to the extreme left, put

$$\theta = -\theta_0;$$

$$\therefore \cos\sqrt{\frac{g}{l}}\,t = -1;$$

$$\therefore t = \pi\sqrt{\frac{l}{g}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

Remark. — In putting $\theta = \sin\theta$ we have introduced an error. If the swing is small, the error is very small. If, for example, $\theta_0 = 5^0$, the time of a swing is in error by $\pi\sqrt{\frac{l}{g}}$ (0.000476) seconds. If $\theta_0 = 1^0$, the error is $\pi\sqrt{\frac{l}{g}}$ (0.00019) seconds.

197. The Seconds Pendulum. — It is evident from the equation

$$t = \pi\sqrt{\frac{l}{g}}$$

that given t, g supposed constant, we may find l. In order to find the length, L, of the pendulum which oscillates once in one second, put $t = 1$ in equation (74).

$$L = \frac{g}{\pi^2}.$$

L is called the length of a seconds pendulum.

198. We give without proof the following formula. If L is the length of a seconds pendulum, and l the length of the

pendulum that beats seconds when it swings through a large angle $2A$,

$$l = L\left(1 - \frac{1}{2}\sin^2\frac{A}{2} - \frac{3}{32}\sin^4\frac{A}{2}\cdots\right).$$

199. Differential Corrections. — Let us consider the following problem. Suppose the length l of the pendulum considered in Art. 196 be changed by an amount Δl, and that g remains constant. Find the amount Δt that t, the time of one oscillation, will change.

From equation (74),

$$t + \Delta t = \frac{\pi}{\sqrt{g}}\left(l^{\frac{1}{2}} + \frac{1}{2}l^{-\frac{1}{2}}\Delta l - \frac{1}{8}l^{-\frac{3}{2}}\overline{\Delta l}^2\cdots\right) \quad . \quad . \quad . \quad (a)$$

If Δl is so small that $\overline{\Delta l}^2$ and all the higher powers of Δl may be neglected in comparison with Δl, and if we subtract equation (74) from equation (a), we obtain

$$\Delta t = \tfrac{1}{2}\frac{\pi}{\sqrt{gl}}\Delta l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

It will be noted that the coefficient of Δl in (b) is the derivative $\frac{dt}{dl}$, obtained by differentiating (74), and further that the operation by which we obtain Δt in (b) is just the method of differentiation provided Δl be chosen so small that the series on the right side of (a) converges rapidly.

This is a special case of a very general problem. It is oftentimes desirable to change the conditions under which a mechanism is working, or to vary the parts of a mechanism itself, so as to change its behavior. In such cases we proceed as we did in the problem just solved. Suppose the motion of a particle or the parts of a mechanism depends upon several things (as for example in the case we are considering, the time of an oscillation of a pendulum depends on its length, l, and the quantity, g) and that we desire to find the change in the behavior of the mechanism by changing any one of the variables

at our command (in this case the quantity, l). We proceed thus: First, find a true relation connecting the quantities $\left(\text{for example, in the case of the pendulum, } t = \pi\sqrt{\frac{l}{g}}\right)$; second, differentiate this relation, using the variable at our command as the independent variable (in our case the length, l).

To illustrate farther: In order to determine the form of the earth, one method adopted is to carry a pendulum of constant length to different parts of the earth, thus *varying g*. Differentiating (74), we obtain

$$dg = -\frac{2g}{t}\,dt.$$

Hence, having observed dt, we may calculate dg. The negative sign in the last equation shows that as g increases t decreases, and conversely.

In many cases all of the elements of the mechanism except one may be kept constant, and the above method applies. But in case more than one quantity vary, we have to use the method of total differentiation. For example, in the problem just considered, suppose that l and g both vary, then

$$dt = \frac{\partial t}{\partial l}\cdot dl + \frac{\partial t}{\partial g}\cdot dg.$$

In this case it is necessary to know a relation between t, l, and g, and to be able to observe the variations of two of these in order to compute the third. This method, as has been said, may be applied to any mechanism, if we can find a relation between the elements which affect its motion.

In order to reduce the error of observation in the cases of the pendulum, just considered, it is customary to observe, instead of the time of a single swing, the number of oscillations that a pendulum makes in a large number of seconds. It follows easily that if $n =$ the number of oscillations that a pendulum makes in N seconds,

$$n = \frac{N}{\pi}\sqrt{\frac{g}{l}}.$$

The student may prove that, l remaining constant,

$$dn = \frac{n}{2g}\,dg.$$

200. **Exercises**

1. A pendulum 3 feet long makes 62.5 vibrations in a minute. Find g. *Ans.* $g = 32.13$.

2. The length of the seconds pendulum in London is 39.139 inches. Find the value of g there. *Ans.* 32.19.

3. A clock which should beat seconds was found to lose 2 minutes a day at a place where $g = 32.2$. How many turns to the right should be given a nut raising the pendulum bob, the screw having 50 threads to the inch?

4. Find the number of vibrations that a pendulum will gain in N seconds by shortening the length of l by a given amount dl, if g is constant.

5. A pendulum clock loses 40 seconds per day (24 hours) when carried from sea level to the top of a mountain. How high is the mountain? (Take the radius of the earth as 4000 miles.)

201. Motion on a Smooth Curve. Centripetal and Centrifugal Force. — Let us now return to the general case considered in Art. 195. Let us suppose the curve is smooth, then since

$$\cos\alpha = -\frac{dy}{ds}, \text{ and } \sin\alpha = +\frac{dx}{ds},$$

equations (70) become

$$\frac{W}{g}\frac{d^2x}{dt^2} = X_1 - R\frac{dy}{ds},$$

$$\frac{W}{g}\frac{d^2y}{dt^2} = Y_1 + R\frac{dx}{ds}.$$

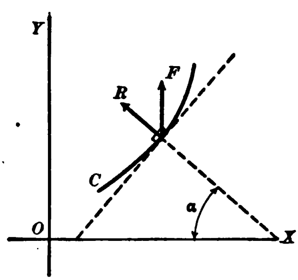


Fig. 119

Multiplying these equations by $\frac{dy}{ds}$ and $\frac{dx}{ds}$ and solving for R, and remembering that

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2=1,$$

we get
$$R=\frac{W}{g}\left(\frac{d^2y}{dt^2}\frac{dx}{ds}-\frac{d^2x}{dt^2}\frac{dy}{ds}\right)+X_1\frac{dy}{ds}-Y_1\frac{dx}{ds}$$

$$=\frac{Wv^2}{g\rho}+X_1\frac{dy}{ds}-Y_1\frac{dx}{ds}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where ρ, by equation (57), Art. 167, is the radius of curvature.

This quantity, $\frac{Wv^2}{g\rho}$, is sometimes called the *centripetal force.* It is the reaction of the curve on the particle due to its motion. Then, since to every action there is an equal and opposite reaction, the reaction of the particle *on the curve due to its motion is equal in magnitude* to $\frac{Wv^2}{g\rho}$; it is normal to the curve and *directed away from* the center of curvature (see Art. 167). This is sometimes called the *centrifugal force.*

From Art. 186 the work of the forces acting on a body is

$$W=\int(Xdx+Ydy)=\frac{1}{2}\frac{W}{g}(v^2-v_0^2),$$

where X and Y are the resolved parts of all the forces acting on the particle. In this case

$$X=X_1-R\frac{dy}{ds}.$$

$$Y=Y_1+R\frac{dx}{ds}.$$

Let us suppose that the only external force acting on the particle is its weight, then

$$X_1=0 \text{ and } Y_1=-W.$$

Then

$$\int(Xdx + Ydy) = \int\left(-R\frac{dy}{ds}dx - Wdy + R\frac{dx}{ds}dy\right) = \int - Wdy;$$

hence, if $v = v_0$ when $y = y_0$,

$$\frac{1}{2}\frac{W}{g}(v^2 - v_0^2) = W(y_0 - y).$$

That is, the velocity of a particle moving on a smooth (plane) curve depends only on the vertical distance $(y_0 - y)$ through which the particle has moved, and is independent of the path in which it moves. This is an illustration of a conservative system of forces. (See Art. 187.) In the case of the pendulum this may be written

$$\tfrac{1}{2}(v^2 - v_0^2) = gl(\cos\theta_0 - \cos\theta).$$

202. Motion on a Rough Curve. — Let us suppose that *the curve is rough;* then the total reaction, R, of the curve on the particle may be resolved into N, the normal pressure, and the friction fN, along the curve, where f is the coefficient of friction; the equations of motion become

$$\frac{W}{g}\frac{d^2x}{dt^2} = X - fN\frac{dx}{ds} - N\frac{dy}{ds};$$

$$\frac{W}{g}\frac{d^2y}{dt^2} = Y - fN\frac{dy}{ds} + N\frac{dx}{ds}.$$

Hence,

$$N = \frac{Wv^2}{g\rho} + X\frac{dy}{ds} - Y\frac{dx}{ds},$$

and

$$R = N\sqrt{1 + f^2}.$$

203. The Conical Pendulum; the Governor. — Suppose a particle P of weight W be attached to one end of a string of length l, the other end of which is fixed at A. Suppose the

particle P is made to describe a horizontal circle PO with uniform velocity. The radius of the circle is $PO = r = l \sin \theta$.

Since the body is describing a horizontal circle, there is no vertical acceleration.

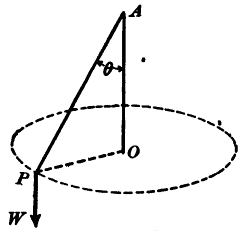


FIG. 120

Let $T =$ tension of the string,

then $T \cos \theta = W$,

and $T \sin \theta =$ centrifugal force,

$$= \frac{Wv^2}{gr}.$$

Hence, $\tan \theta = \dfrac{v^2}{gr}.$

For a given velocity, therefore, the inclination of the string to the vertical is determined. It is evident that the greater v is, the greater θ is.

If now the angular velocity $= \omega$,

$$T \sin \theta = \frac{Wr^2\omega^2}{gr} = \frac{Wr\omega^2}{g} = \frac{Wl}{g} \sin \theta \omega^2.$$

$$\therefore\ T = \frac{W}{g} l\omega^2.$$

REMARK. — This is the mechanics of the old governor for steam engines.

204. Exercises

1. A weight W is placed at P on a rough horizontal bar, OP, which is made to revolve around a vertical axis through O, with an angular velocity ω. If the coefficient of friction $= f$, find the angular velocity ω when the weight is about to slide. *Ans.* $\omega^2 = \dfrac{fg}{OP}.$

2. Show that if the earth rotates and is not rigid it must be flattened at the poles.

3. Show that the time of revolution of a conical pendulum is the same as the time of oscillation of a simple pendulum of length $l \cos \theta$.

4. A railway train is going smoothly along a circular curve of 1500 feet radius at the rate of 30 miles an hour. Find at what angle a lamp hanging in one of the carriages, swinging freely, will be inclined to the vertical. *Ans.* 2° 18′ almost.

5. If this lamp be set swinging through a small angle, will it vibrate in less, in greater, or in the same time than it would if set swinging through the same angle when the car is at rest?

6. A pendulum bob weighs 5 pounds. It is pulled aside through an angle of 30° and let go. What will be the tension on the string when the bob is at its lowest point?

Ans. $T = 6.34$ pounds.

7. A pendulum bob moves through its lowest point with a velocity v_0. Find a value of v_0 such that the pendulum will describe a complete circle:

(1) If CP is a string;

(2) If CP is a stiff, weightless rod.

Ans. (1) $v_0^2 = 3\,ag$;

(2) $v_0^2 = 2\,ag +$ any small quantity.

8. A pendulum bob weighing 2 pounds is drawn aside through an angle of 45° and let go. Find the tension on the string when it makes an angle of $-45°$ with the vertical.

9. A coal wringer recently invented consists of a hollow circular cylinder with a radius of 2 feet, rotating around its axis, which is vertical. If this cylinder is rotating 300 times per minute, with what force will a piece of coal weighing 1 pound press against the inner surface of the cylinder?

10. A train is traveling with a velocity v on a circular track whose radius of curvature $= r$; the distance between the rail is a. Show that the train will travel as safely as on a straight track if the outer rail is raised a distance h, where

$$\frac{h}{\sqrt{a^2 - h^2}} = \frac{v^2}{rg}.$$

In what units must the various quantities in this equation be expressed?

CHAPTER XIII

IMPULSE; COLLISION OF SPHERES

205. We shall in the present chapter consider the behavior of bodies in collision. The single instantaneous stroke of a body in motion against another body either in motion or at rest we shall call an *impact*. The behavior of a body after the impact depends, among other things, on its elasticity. There are numerous illustrations of the impact of bodies which are either perfectly or imperfectly elastic. Billiard balls are nearly perfectly elastic. The kinetics of two impinging billiard balls (perfectly spherical and perfectly elastic) is the same as that of the collision of two molecules of a gas, and is at the foundation of the great and fruitful theory known as the Kinetic Theory of Gases, by which we account for the pressure of gases on the walls of a vessel which contains it, as for example the pressure of steam in a boiler. The resistance of air to a moving train results from the collision of the molecules of air with the train. The resistance of water to a ship in motion, the pressure of wind on sails, or on the vane of a windmill, the driving of piles with a pile driver, are additional instances of collision.

206. In Chapter XI we considered the quantity $\int_{s_0}^{s} F ds$, which we defined as the work done by F in moving a particle along a path whose length is $s - s_0$. In this chapter, we shall consider the quantity $\int_{t_0}^{t} F dt$.

From Newton's Second Law of Motion,

$$\frac{d}{dt}\left(\frac{Wv}{g}\right) = F. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence, if we assume $v = v_0$ when $t = t_0$,

$$\frac{Wv}{g} - \frac{Wv_0}{g} = \int_{t_0}^{t} Fdt. \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

If F acts in a constant direction during the interval $t - t_0$, the expression $\int_{t_0}^{t} Fdt$ is called the *impulse* of the force F during the interval $t - t_0$, $t - t_0$ being very small.

Accordingly we do not attempt to evaluate the right side of (76) by the methods of the integral calculus. We shall now prove the lemma.

207. *Lemma.* — Suppose a system of particles each of which is moving in the same direction. Let us take this direction as the x-axis. Let the weight of these particles be, respectively,

$$W_1, \ W_2 \cdots$$

and their positions at any time be, respectively,

$$x_1, \ x_2 \cdots$$

Let $\bar{x} = x$-coördinate of their center of gravity, then (Art. 93),

$$\bar{x} = \frac{W_1x_1 + W_2x_2 + \cdots}{W_1 + W_2 + \cdots},$$

or

$$W\bar{x} = W_1x_1 + W_2x_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

where

$$W = W_1 + W_2 + \cdots.$$

Hence, differentiating (a), we get

$$\frac{W}{g}\frac{d\bar{x}}{dt} = \frac{W_1}{g}\frac{dx_1}{dt} + \frac{W}{g} \cdot \frac{dx_2}{dt} + \cdots.$$

Hence the sum of the momenta of any system of particles moving in a given direction is at any time equal to the momentum of a body whose mass equals the sum of all the masses, and which is concentrated at the center of gravity of the system. Since, in the problems that we shall consider, the above conditions are fulfilled, we may substitute for our system of particles a particle at the center of gravity of the system.

208. Impact of Spheres. — We shall consider, only, the impact of spheres which are not rotating. The problem is indeterminate if the spheres are rigid; and, since there are no rigid bodies in nature, we shall assume that the bodies are deformed under the collision and that they are not deformed to such an

extent that they will not regain their shape. For the sake of convenience we shall divide the time in which the bodies are in contact into three periods.

1. The period in which the bodies are being compressed.
2. The instant of greatest compression.
3. The period of restitution, that is, the period in which the spheres tend to regain their shape.

Each of these periods is infinitely short. The compression and restitution both act as a retarding force on the impinging sphere, and as an accelerating force on the sphere impinged upon.

209. Direct Impact. — For the moment we shall assume that the impact is direct, *i.e.* that the motion of both spheres is along the line joining their centers. The problem may then be stated as follows:

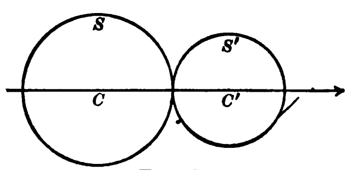


FIG. 121

Let S and S' be two homogeneous spheres whose centers are at C and C', moving in the direction CC'. Let the weight of S be W and S' be W'. Let S impinge on S', assuming the direction in which S is moving as positive. Let

u = the velocity with which S is moving before impact;

u' = the velocity with which S' is moving before impact;

v = the velocity with which S is moving after impact;

v' = the velocity with which S' is moving after impact;

To find the momentum of the system, and the velocities v and v'.

In equation (76) let F, which is variable, be the force of reaction between the spheres. We shall have, considering the motion of S,

$$\frac{W}{g}(u-v)=\int_{t_0}^{t} F dt.$$

Now by the Third Law of Motion, the reaction on S' is equal and opposite to that on S. Hence, considering the motion of S',

$$-\int_{t_0}^{t} F dt = \frac{W'}{g}(u' - v');$$

$$\therefore \frac{W}{g}(u - v) = \frac{-W'}{g}(u' - v').$$

$$\therefore \frac{Wu}{g} + \frac{W'u'}{g} = \frac{Wv}{g} + \frac{W'v'}{g}. \quad . \quad . \quad . \quad . \quad (77)$$

That is, the momentum of the system is the same after impact as it was before impact.

210. Coefficient of Elasticity. — In general, the Impulse of Restitution is less than that of Compression. If there is no force of Restitution, we say that the spheres are inelastic. If the Impulse of Restitution equals the Impulse of Compression, the bodies are said to be perfectly elastic. For all other cases, the bodies are said to be imperfectly elastic. The ratio of the Impulse of Restitution to that of Compression is called the coefficient of elasticity, which we shall denote by e.

211. The Velocity after Impact. — At the instant of greatest compression the spheres have the same velocity.

Let $\qquad V =$ this common velocity.

Then $\quad \frac{W}{g}(u - V) =$ impulse of compression on W;

$$\frac{W}{g}(V - v) = \text{impulse of restitution on } W;$$

$$\therefore V - v = e(u - V).$$

Similarly,

$$v' - V = e(V - u').$$

$$\therefore v' - v = e(u - u'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

Substituting from (77) and solving for v, we get

$$v = \frac{Wu + W'u' - W'e(u - u')}{W + W'}. \quad . \quad . \quad (79)$$

Similarly,
$$v' = \frac{Wu + W'u' + We(u - u')}{W + W'}. \quad \quad (80)$$

If the spheres are, before impact, moving in opposite directions, u' is negative. If, before impact, S' is at rest, $u' = 0$.

COROLLARY. *The reader will observe that the reasoning will apply to a sphere striking a plane perpendicularly. For a plane may be considered as a sphere of infinite radius, and, if the plane be fixed, considered as having an infinite mass. In fact, the reasoning applies to the collision of any two bodies which are not rotating, provided they are moving in the line joining their centers of gravity, and this line is normal to both surfaces at the point of contact.*

212. Exercises

1. Two glass balls ($e = 1$) weighing 16 oz. and 12 oz., respectively, move in the same line with velocities of 5 ft. and 4 ft. per second. What are their velocities after impact?

(a) If their original velocities were of the same sense?

(b) If their original velocities were of opposite sense?

Ans. (a) $4\frac{1}{7}$, $5\frac{1}{7}$; (b) $-2\frac{5}{7}$, $6\frac{2}{7}$.

2. If a perfectly elastic ball strike another of equal mass at rest, find the velocities after impact.

3. A ball weighing 10 pounds moving with a speed of 50 feet per second overtakes a ball weighing 25 pounds, moving with a speed of 20 feet per second and in the same direction. If $e = \frac{1}{5}$, find the velocities after impact.

4. A ball is dropped from a height h above a horizontal plane to the plane. Find (a) the velocity, v, with which it leaves the plane; (b) the height, h_1, to which it will rise if the coefficient of elasticity $= e$. *Ans.* (a) $v = e\sqrt{2gh}$; (b) $h_1 = e^2 h$.

5. The ball of exercise 4 will rebound repeatedly; find how high it will rise on the tenth rebound.

6. If in exercise 5, $e = \frac{1}{8}$ and $h = 20$, find the distance it will have traveled when it has struck the plane the fifth time.

Ans. 25 feet (about).

7. Devise a method for finding e of a golf ball.

8. If a ball weighing 5 pounds moving with a speed of 4 feet per second meets another ball weighing 20 pounds, moving with a speed of one foot per second, find the velocity of the bodies after impact if $e = 0$.

9. A number of equal, perfectly elastic balls are placed in contact with their centers in a straight line. An equal ball impinges with a velocity u along this line on the first ball of the row. Show that the last ball of the row will move with the velocity u while all the others will remain at rest.

213. Oblique Impact. — We shall, as in Art. 208, assume that the bodies are spheres and not rotating, and in addition that they are smooth, but that the spheres are not moving in the line joining their centers.

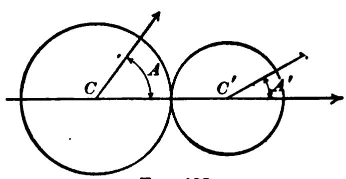


Fig. 122

Let the weights of the spheres be W and W'; let their centers be C and C'. Let their velocities *before impact* be respectively u and u', making the angles A and A' with the line CC'. Let their velocities after impact be v and v', making the angles B and B' respectively with the line CC'.

Resolve the velocities along the line CC' and perpendicularly to CC'.

Before impact the resolved part of the velocities *along* CC' are respectively $u \cos A$ and $u' \cos A'$; and the resolved parts perpendicular to CC' are $u \sin A$ and $u' \sin A'$.

After impact the resolved parts of the velocities *along* CC' are respectively $v \cos B$ and $v' \cos B'$ and perpendicular to CC' are respectively $v \sin B$ and $v' \sin B'$. Now, since the spheres are smooth, the reactions of the spheres are along the

line CC'. Hence the velocities of the spheres perpendicular to CC' are unchanged.

Therefore,
$$\left.\begin{aligned} u \sin A &= v \sin B; \\ u' \sin A' &= v' \sin B'. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (a)$$

For the resolved part of the velocities along the line CC' the conditions are just the same as in the case of direct impact. Hence from equations (79) and (80), Art. 211, we get

$$v \cos B = \frac{Wu \cos A + W'u' \cos A' - W'e(u \cos A - u' \cos A')}{W + W'}, \quad (81)$$

and

$$v' \cos B' = \frac{Wu \cos A + W'u' \cos A' + We(u \cos A - u' \cos A')}{W + W'}. \quad (82)$$

Equations (a), (81), and (82) involve four unknowns, viz. v, v', B, and B', and hence these equations completely solve our problem.

214. The particular case of the oblique impact of a smooth sphere on a smooth fixed plane deserves special consideration.

In this case $u' = 0$, $v' = 0$, and $W' =$ infinity (Art. 211). Dividing both numerator and denominator by W' and putting

$$W' = \infty, \text{ we get}$$

$$v \cos B = - eu \cos A.$$

Also,
$$v \sin B = u \sin A;$$

$$\therefore \cot B = - e \cot A,$$

or
$$\tan A = - e \tan B.$$

This equation shows that if $A < 90°$, then $B > 90°$ and conversely. In case $e = 1$ the angle of *incidence* equals the angle of *reflection.*

215. There are numerous problems and generalizations of the theory of impact that deserve attention, but which carry us beyond the limit we have set for ourselves. The whole

theory of gases is only the theory of impact of the molecules of the gas. Resistance to motion of trains in the air, and of steamships in water, is merely the impact of the train on air particles and the ship on water particles.

The problems become complicated at once, but not impossible if the bodies are not smooth or if they are rotating — a familiar fact to one who "cuts" tennis balls, in order to change the direction of the bound. This latter problem is discussed fully in Williamson's and Tarleton's *Dynamics*.

216. **Exercises**

1. A sphere weighing 5 pounds strikes a horizontal plane at an angle of 45° with a velocity of 1 foot per second. If $e = .1$, find the velocity and the angle with which it will rebound.

2. A baseball weighing $5\frac{1}{4}$ oz. moving with a velocity of 100 ft. per second is struck by a bat (assume the effect to be the same as if struck by a sphere) in a direction at right angles to its line of motion. Find the momentum imparted by the blow if it deflects the ball through an angle of 60°.

3. A ball of weight W impinges on a ball of weight W' at rest; find the conditions that the directions of motion of the impinging ball before and after impact may be at right angles.

Ans. $\tan^2 A = \dfrac{W'e - W}{W' + W}$.

4. A ball impinges on an equal ball at rest; the angle between the old and the new directions of the impinging ball is 60°. Find the velocity of impinging ball after impact, if $e = 1$.

Ans. $u \sin 30°$.

217. Change in the Kinetic Energy by Direct Impact. — If we obtain $\dfrac{Wv^2}{g}$ and $\dfrac{W'v'^2}{g}$ from the equations (77), (79), and (80), we shall find after a rather tedious but elementary reduction

$$\tfrac{1}{2}Wv^2 + \tfrac{1}{2}W'v'^2 = \tfrac{1}{2}Wu^2 + \tfrac{1}{2}W'u'^2 - \tfrac{1}{2}(1-e^2)\frac{WW'}{W+W'}(u-u')^2. \quad (a)$$

Now

$\frac{1}{2}\frac{Wv^2}{g}+\frac{1}{2}\frac{W'v'^2}{g}$ and $\frac{1}{2}\frac{Wu^2}{g}+\frac{1}{2}\frac{W'u'^2}{g}$ are respectively the kinetic energy of the system after and before impact. Hence the kinetic energy is less after impact than before unless $e = 1$; for the last term of (a) is positive, since $e < 1$ in all bodies.

It is common to say that kinetic energy is "lost." This is true so far only as the kinetic energy of the particular system is concerned. This "lost" kinetic energy has been used in producing sound or heat, or in tearing away obstruction, as in the case of driving a nail, or in some other form of doing work.

218. By the theory of Chapter XI we may find the work done by one body impinging on another.

For example, a pile driver weighing W pounds, falling from rest through a height h, will drive a pile against a constant resistance R, through a distance s, where, neglecting "lost" energy,

$$\frac{1}{2}\frac{Wv^2}{g} = Rs,$$

or, we may write, simply,

$$Wh = Rs.$$

In the case of a projectile thrown from a gun, the recoil of the gun can be found, either in terms of the distance it will recede against a resistance, or the velocity with which it will recede.

219. **Exercise**

1. At the Chestnut Street Bridge, Phila., piles were driven until they sank only $\frac{3}{4}$ of an inch under each blow from a 1200-pound hammer falling 20 feet. What was the resistance overcome by the blow?

CHAPTER XIV

THE MOMENT OF INERTIA

220. Definition. — Let us consider a system of particles P_1, $P_2 \cdots P_n$, whose masses* are m_1, m_2, $\cdots m_n$ respectively, and which are at a perpendicular distance r_1, $r_2 \cdots$ respectively from a given line l. Then the Moment of Inertia, I, of this system with respect to the line l is defined by the equation

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots m_n r_n^2 \quad . \; . \; . \; . \; . \quad (83)$$
$$= \Sigma m r^2. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (83)$$

The expression $I = \Sigma m r^2$ occurs frequently in equations relating to the mechanics of rigid bodies; notably in determining the behavior of rotating bodies. It is a mathematical fiction rather than a physical conception. That is, it is a series of terms occurring so frequently in equations of the kinetics of bodies, and a modified form of it in the mechanics of the flexure of beams, that it has received the name moment of inertia — a misnomer in fact, and it is always designated by I. In order therefore to clear the way for the solution of the problems in succeeding chapters, we shall deduce certain theorems and find the moment of inertia of certain bodies. The problem is usually one of integration, similar to that of the center of gravity. For a body may be considered as a collection of particles. If the body is continuous we may put $m_i = dm$, and replace the summation sign by the integral sign. Equation (83) then becomes

$$I = \int r^2 dm. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (84)$$

* In this chapter we have written $m = \text{mass} = \frac{W}{g}$ (Art. 142).

As in Art. 95, $dm = \frac{\rho dV}{g}$, where ρ is the weight per cubic unit and dV the differential volume. (See Arts. 96 and 97.) The line, l, with respect to which the moment of inertia is taken, is called the *axis*. We shall sometimes denote the moment of inertia of a body with respect to such an axis, l, by I_l. If the axis passes through the center of gravity of the body, it is called a *gravity* axis.

In what follows, unless otherwise stated, we shall consider that all bodies are homogeneous, that all areas are of uniform thickness, and that all lines have uniform cross sections.

221. An Important Theorem. — *The moment of inertia I_l of a body about any axis l = the moment of inertia, I_G, of the body about a parallel gravity axis + the mass of the body multiplied by the square of the distance between these axes.*

Proof. — Take a plane through G, the center of gravity of the body, perpendicular to the given axis as the xy-plane. Let l cut this plane in O, and a line drawn parallel to l through dm, any differential mass of the body, cut the plane in P.

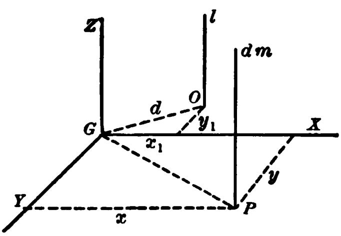


Fig. 123

Let the coördinates of O be x_1, y_1 when referred to G; and the coördinates of P be x, y, and x', y' when referred to G and O respectively; let $GO = d$.

Then, $x' = x - x_1$,

$y' = y - y_1$,

$$\Sigma m(x'^2 + y'^2) = \Sigma m(x^2 + y^2) + \Sigma m(x_1^2 + y_1^2) - 2\,\Sigma mxx_1 - 2\,\Sigma myy_1.$$

That is,
$$I_l = I_G + Md^2, \quad \cdots\cdots \quad (85)$$
where
$$M = \Sigma m;$$
for, $\Sigma mxx_1 = x_1\Sigma mx = x_1M\bar{x}$ (equation 15′, Art. 95) $= 0$ (since G is the origin). And similarly $\Sigma myy_1 = 0$.

It is hardly necessary to point out that the proof is perfectly general, and as a special case, if the body is plane, the axis may be either perpendicular or parallel to the plane of the body.

It is evident from (85) that the moment of inertia of a body is less for a given gravity axis than for any other axis parallel to it.

COROLLARY. *If l and l' be two parallel lines distant respectively d and d' from a parallel line through the center of gravity of a body of mass M, then*

$$I_l - Md^2 = I_G = I_l' - Md'^2. \quad . \quad . \quad . \quad . \quad (86)$$

222. The Radius of Gyration. — The quantity $\sqrt{\frac{I}{M}}$ is defined as the *radius of gyration* of a body whose moment of inertia is I and whose mass is M; it is usually denoted by k; that is,

$$k = \sqrt{\frac{I}{M}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

$$I = Mk^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87')$$

From (87′) it is evident that the moment of inertia of the body is equal to the moment of inertia of a body of equal mass concentrated at a distance k from the axis, or is equal to the moment of inertia of a particle of mass M at a distance k from the axis.

REMARK. — The student should keep in mind that the radius of gyration, like the moment of inertia, is a mathematical fiction rather than a physical conception (see Art. 220, which applies with equal force here).

223. We shall now compute the moment of inertia of certain bodies, the motions of which are considered in later chapters.

224. To find the moment of inertia of a homogeneous straight line with respect to an axis through one end and perpendicular to it.

Take O the end of the line through which the axis passes as origin, and the line as axis of x. Then,

$$dm = \frac{\rho}{g} A dx, \text{ where } A \text{ is the area of the cross section.}$$

$$I = \int_0^a A\frac{\rho}{g} dx \cdot x^2 = A\frac{\rho}{g}\frac{x^3}{3}\Big]_0^a = \frac{Ma^2}{3},$$

where $M =$ mass, and $a =$ length of the line.

COROLLARY. Since (Art. 90) a plane surface may be built up by laying side by side an infinite number of thin rods, it follows, as the student should show, that the moment of inertia of the rectangle $ABCD$ about AB (if $AB = a$, and $AC = b$, the mass of rectangle $= M$) is

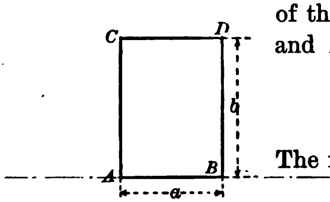


FIG. 124

$$I_{AB} = \frac{M \cdot b^2}{3}.$$

The radius of gyration

$$k_{AB} = \frac{b}{\sqrt{3}}.$$

The moment of inertia of the rectangle $ABCD$ about AC is

$$I_{AC} = \frac{Ma^2}{3};$$

$$k_{AC} = \frac{a}{\sqrt{3}}.$$

Exercise.—Find moment of inertia of the triangle ABD, Fig. 124, about the axis AB.

225. Plane Thin Plates. Polar Moment of Inertia of Plane Thin Plates.—In this case,

$$dm = \frac{\rho}{g} \cdot t \cdot dA,$$

where t is the thickness of the plate and dA is a differential area.

The moment of inertia, I_x, of a plane thin plate about the x-axis is, by definition,

$$I_x = \Sigma my^2.$$

The moment of inertia, I_y, of a plane thin plate about the y-axis is

$$I_y = \Sigma mx^2.$$

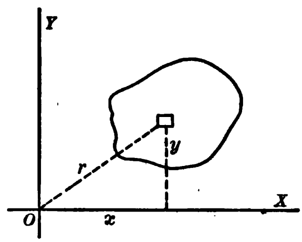


FIG. 125

The moment of inertia of a plane lamina about an axis perpendicular to the plane is usually called the *Polar Moment of Inertia* and designated by I_p. If the axis passes through the origin

$$I_p = \Sigma mr^2 = \Sigma m(x^2 + y^2) = \Sigma mx^2 + \Sigma my^2,$$

or $$I_p = I_x + I_y. \quad \text{(See Fig. 124.)} \quad \ldots \ldots \quad (88)$$

COROLLARY. The polar moment of inertia of a rectangle (a by b) about an axis through one corner is

$$I_p = \frac{M}{3}(a^2 + b^2),$$

where $$M = \frac{\rho}{g}\, ab\,.\, t = \text{mass of the rectangle.}$$

226. Moment of Inertia of a Circumference. — The moment of inertia of a *circumference* of radius r (1) about an axis perpendicular to its plane through the center is

$$I_p = Mr^2 \text{ (definition); and}$$

(2) about a diameter is

$$I_D = \frac{Mr^2}{2} \text{ (from (88)).}$$

227. Moment of Inertia of a Circular Area. — (1) To find the moment of inertia of a *circular area* of radius a about an *axis perpendicular to the plane of the area,* through the center of gravity, we proceed as follows:

Take as elementary mass a concentric ring of radius r and thickness t.

Then $dm = \frac{\rho}{g} t \, 2\pi r \cdot dr$. Let dI be its polar moment of inertia; then

$$dI = \frac{\rho}{g} \cdot t \, 2\pi r \cdot dr \cdot r^2;$$

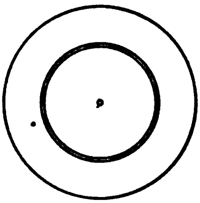


FIG. 126

and the polar moment of inertia of the entire plate is

$$I_p = \int_0^a 2\pi \frac{\rho}{g} \cdot tr^3 dr = \frac{Ma^2}{2}.$$

(2) The moment of inertia of a *circular area about a diameter* I_D is evidently (equation (88)),

$$I_D = \frac{I_p}{2} = \frac{Ma^2}{4}.$$

(3) The moment of inertia of a *circular area* I_t *about an axis perpendicular to the plane and passing through the circumference is*

$$I_t = I_p + Ma^2 = \frac{3}{2} Ma^2.$$

228. Right Circular Cylinder. — To find the moment of inertia of a *right circular cylinder* about an axis perpendicular to the axis of the cylinder, and intersecting that axis. Let the axis AB be a distance d from the nearest base. We may think of the cylinder as made up of thin circular plates the planes of which are perpendicular to the axis of the cylinder. Choose as our elementary volume one of these plates at a distance x from the axis, and of thickness dx. The moment of inertia of this plate about its diameter is (Art. 227 (2)).

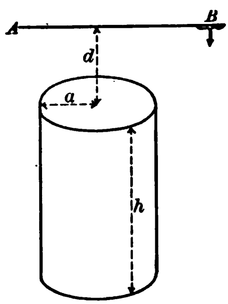


FIG. 127

$$\pi \cdot \frac{\rho}{g} \frac{a^2 \cdot dx \cdot a^2}{4}.$$

Hence the moment of inertia of this plate with regard to the axis AB is

$$dI = \frac{\rho}{g} \pi a^2 dx \left(\frac{a^2}{4} + x^2 \right).$$

Hence the moment of inertia of the entire cylinder about axis AB is,

$$I_{AB} = \pi \frac{\rho}{g} \int_d^{d+h} a^2\left(\frac{a^2}{4} + x^2\right)dx$$

$$= M\left(\frac{a^2}{4} + \frac{h^2 + 3\,d^2 + 3\,dh}{3}\right);$$

where M is the mass of the cylinder.

If AB is a diameter of the base

$$I = M\left(\frac{a^2}{4} + \frac{h^2}{3}\right).$$

229. Solids of Revolution. — Modifying slightly the method of the preceding article, we may find the moment of inertia of any solid of revolution about a line l, perpendicular to the axis of the surface. Suppose an area bounded by the x-axis, a curve whose equation is $y = f(x)$, and an ordinate is revolved about the x-axis. Then the solid may be thought of as being made up of circular

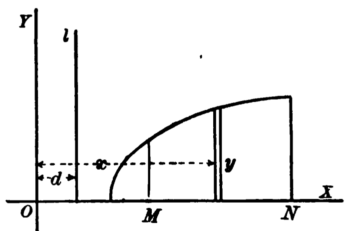


FIG. 128

laminæ each perpendicular to the x-axis. The moment of inertia of each lamina about one of its diameters is $\frac{dmy^2}{4}$ (Art. 227). If l is parallel to the y-axis at a distance d from it, the moment of inertia of any lamina about this axis is

$$dI = \frac{\rho}{g}\,\pi y^2 \cdot dx\left(\frac{y^2}{4} + (x - d)^2\right).$$

Hence, $$I = \int_{x_1}^{x_2} \frac{\rho}{g}\,\pi y^2\left(\frac{y^2}{4} + (x - d)^2\right)dx;$$

where $OM = X_1$ and $ON = X_2$.

The y under the integral sign may be expressed in terms of x from the equation of the curve.

230. Compound Figures. — Many times it is required to find the moment of inertia of figures which yield more readily to elementary methods than to the methods of the calculus. It is evident that the moment of inertia of a body about an axis is equal to the sum of the moments of inertia of its component parts, with respect to that axis.

Also in this connection it is worth while to note that the moment of inertia of a figure is not altered if we move an element or combination of elements in a direction parallel to the axis, about which the moment of inertia is being obtained.

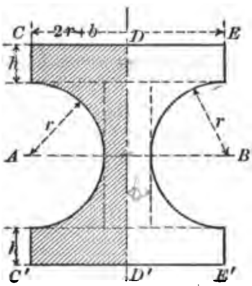


Fig. 129

To illustrate these principles, let us find the moment of inertia about the axis AB of the plane area shown in Fig. 129, (1) using the relation

$$I_l = I_G + Md^2;$$

(2) using the second principle which we have just stated.

(1) Let us divide the figure into three rectangles, two being $(2r+b)\cdot h$, and one $b\cdot 2r$, and the four areas, each of which is the part of a square $r\cdot r$ from which the quadrant $\frac{\pi r^2}{4}$ has been taken. In order to simplify the expressions we shall assume $\frac{\rho}{g}t = 1$, where t is the thickness of the lamina.

Then the moment of inertia of the rectangle $(2r+b)\cdot h$ about an axis through its center of gravity parallel to AB is $\frac{1}{12}(2r+b)h^3$; and the moment of inertia of this rectangle about the axis AB is

$$\frac{1}{12}(2r+b)h^3 + (2r+b)h\cdot\left(\frac{h}{2}+r\right)^2.$$

The moment of inertia of the rectangle $b\cdot 2r$ about AB is

$$= \tfrac{1}{12}\,b\,(2r)^3.$$

The moment of inertia of each of the parts outside the quadrant is

$$=\frac{1}{3}r^4-\frac{\pi r^4}{16}.$$

Then I of the whole figure about AB is

$$I_{AB}=2\left[\frac{1}{12}(2r+b)\,h^3+h(2r+b)\left(\frac{h}{2}+r\right)^2\right]$$

$$+\frac{1}{12}b(2r)^3+4\left[\frac{1}{3}r^4-\frac{\pi r^4}{16}\right]=\frac{1}{12}(2r+b)(2r+2h)^3-\frac{\pi r^4}{4}.$$

(2) If, instead, we divide Fig. 129 by a vertical line DD' bisecting the base, and move the shaded part parallel to the axis AB until the point C coincides with E and C' with E', we have the plane in the form shown in Fig. 130, of which I_{AB} is evidently

$$I_{AB}=\frac{1}{12}(2r+b)(2r+2h)^3-\frac{\pi r^4}{4}.$$

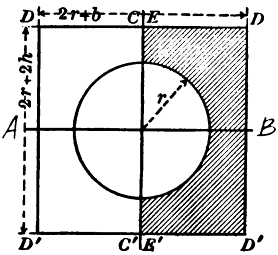
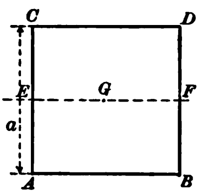


FIG. 130

231. **Exercises**

1. Find the moment of inertia of the square $ABCD$

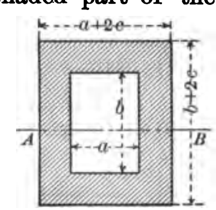


(a) about AB; *Ans.* $\frac{Ma^2}{3}$.

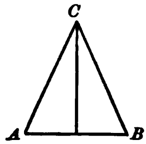
(b) about EF, through the center of gravity parallel to AB. Ans. $\frac{M.a^3}{12}=\frac{AB.a^3}{12}$

(c) Find the polar moment of inertia of the square about a line through the center of gravity of the square.

2. Find the moment of inertia of the shaded part of the figure about an axis AB through the center of gravity of the figure.



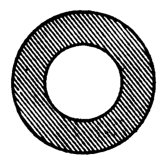
3. Show by a sketch and the principles of Art. 230 that the moment of inertia of the figure in exercise 2 is the same as that of an I-beam.



4. Find the moment of inertia of the triangle ABC, of base b and altitude a, about AB.

Ans. $\dfrac{Ma^2}{6}$.

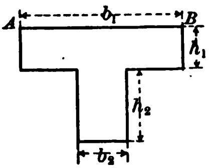
5. Find the moment of inertia of a flat circular ring whose external radius is r_1 and internal radius r_2



(*a*) about a diameter;

(*b*) about an axis perpendicular to the plane and through the center.

6. Find the moment of inertia of the ellipse $\dfrac{x^2}{25} + y^2 = 1$, (*a*) about the x-axis; (*b*) about the y-axis.



7. Find the moment of interia of the cross section of a T-beam with dimensions as shown in the figure.

8. Find the moment of inertia of a right circular cylinder about an element.

Ans. $\frac{3}{2} Mr^2$.

9. Find the moment of inertia of the frustum of a right circular cone, the radius of its upper and lower base being respectively 1 inch and 3 inches, the distance between the bases being 1 foot:

(*a*) about a diameter of the upper base;

(*b*) about the axis of the frustum.

10. Find the moment of inertia of a hollow right cylinder of cast iron 2 feet high, a cross section of which is the ring of exercise 5, in which $r_1 = 12$ inches, and $r_2 = 6$ inches,

(*a*) about an axis perpendicular to, and intersecting the axis of the cylinder, at a distance of 2 feet from the nearest base; and

(*b*) about an axis which is one of the diameters of the base.

Ans. (*a*) $I = 636$.

NOTE. — Cast iron weighs 450 pounds per cubic foot.

11. Find the moment of inertia of a right circular cone, radius of base r, and altitude h, about a line perpendicular to the axis of the cone,

(*a*) through the vertex;
(*b*) through the center of gravity of the cone;
(*c*) coinciding with a diameter of the base.

12. Find the moment of inertia of a sphere of radius r about a diameter. *Ans.* $I = M\frac{2}{5}r^2$.

13. Find the moment of inertia of a sphere of radius r about a tangent line. *Ans.* $I = M\frac{7}{5}r^2$.

14. Find the moment of inertia with respect to the y-axis of a solid generated by revolving around the x-axis the area, bounded by $y^2 = 8x$, the x-axis, and an ordinate whose abscissa is 2.

15. Find the moment of inertia of a rectangular parallelopipedon with a square base, $b \cdot b$, and height c, about an axis AB lying in one base passing through its center of gravity, and perpendicular to one of its sides.

16. Find the moment of inertia of the cone in exercise 11 about its geometric axis. *Ans.* $\frac{3}{10}Mr^2$.

232. The Moment of Inertia of Plane Areas. — In the common theory of the flexure of beams and columns, the expression $\int r^2 dA$ frequently occurs, where dA is a differential area, and r, its distance from a given line or axis. This expression is called the moment of inertia of the plane area about that axis. We often wish to find the moment of inertia, in this sense, of a plane cross section of a beam or column — not the moment of inertia of a thin lamina but of an area. This is a special case of Art. 220, equation (84), in which we substitute dA for dm, and integrate.

233. Exercises

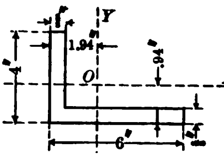
1. Find the moment of inertia of the plane areas described in exercises 1, 2, 4, 5, and 6 (Art. 231).

Ans. 1. (a) $=\frac{1}{3}a^4$; (b) $=\frac{1}{12}a^4$; (c) $=\frac{1}{6}a^4$.

2. $\frac{1}{12}(a+2c)(b+2c)^3-\frac{1}{12}ab^3$.

4. $\frac{1}{12}ba^3$.

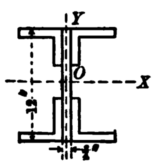
5. (a) $\frac{\pi}{4}(r_1^4-r_2^4)$; (b) $\frac{\pi}{2}(r_1^4-r_2^4)$.



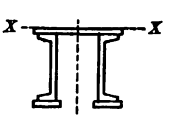
2. Find the moment of inertia of the $6'' \times 4'' \times \frac{3}{8}''$ angle shown in sketch about each of the coördinate axes. O is the center of gravity of the section.

Ans. $I_x=4.9$ in.^{4}, $I_y=13.5$ in.^{4}*

3. A column is composed of 4 angles each $4'' \times 4'' \times \frac{3}{8}''$, and a plate $12'' \times \frac{3}{4}''$. The distance of the center of gravity of the angle from the back of the angle is $1.14''$. First, find the moment of inertia of a single angle about a gravity axis parallel to the x-axis. Second, find the moment of inertia of the entire section about the x- and y-axes.

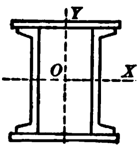


4. The cross section of an end post is made up of a plate $19'' \times \frac{3}{8}''$, two $15''$-channels, each with an area of 9.90 square inches and two flats $4'' \times \frac{3}{4}''$. If the moment of inertia of each of the channels about a gravity axis perpendicular to the web is 312.6 in.^{4}, what is the moment of inertia of the entire section about the x-axis? The center of gravity of the section is 7.65 inches from the x-axis. What is the moment of inertia about a gravity axis parallel to the x-axis?



* This notation means that the linear dimensions are all given in inches; and if we use the "conversion factors" in the dimensional equations (Art. 140) that the lengths enter to the fourth degree.

5. A column is composed of two 15″-channels, weighing 33 pounds per linear foot and two plates $16'' \times \frac{3}{8}''$. Find the moment of inertia of the section about the x-axis; also find how far the channels must be spaced back to back that the moment of inertia about the y-axis shall equal that about the x-axis.



Data: The moment of inertia of a single channel about gravity axes parallel and perpendicular to the web are 8.2 in.^{4} and 312.6 in.^{4} respectively. Distance of the center of gravity from back of channel = 0.79 inches. The x- and y-axes are gravity axes.

234. Problem. — *Given the moment of inertia of a plane area about two axes at right angles to each other and in the plane of the area; to find the moment of inertia of the area about any axis through their intersection and lying in their plane.*

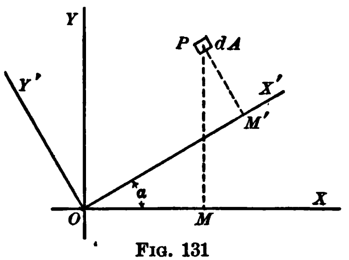


FIG. 131

Let the coördinates of P, the position of any elementary area dA be x, y, referred to the axes, OX and OY. To find the moment of inertia of the area with respect to OX', any line through O making an angle, α, with OX. Draw PM' perpendicular to OX'. Let $PM' = y'$ (say). Then,

$$I_{OX'} = \int y'^2 \, dA.$$

By coördinate geometry,

$$y' = y \cos \alpha - x \sin \alpha.$$

$$I_{OX'} = \int (y \cos \alpha - x \sin \alpha)^2 \, dA,$$

$$= \cos^2 \alpha \int y^2 \, dA + \sin^2 \alpha \int x^2 \, dA - \sin 2\alpha \int xy dA,$$

$$= I_x \cos^2 \alpha + I_y \sin^2 \alpha - \sin 2\alpha \int xy dA. \quad \quad (89)$$

235. Product of Inertia of a Plane Area. — If we multiply each elementary area dA by the product of its coördinates, and sum the products, we obtain the expression $\int xydA$ which is called the *product of inertia* of the area with respect to the coördinate axes. The same thing stated more generally is, if dA is any elementary area and r_1, r_2 its perpendicular distances from two lines perpendicular to each other, then $\int r_1 r_2 dA$ is called the product of inertia of the body with respect to those lines.

236. If x', y' are the coördinates of dA with respect to the axes OX', OY' (Fig. 130), we have

$$x' = x \cos \alpha + y \sin \alpha,$$
$$y' = y \cos \alpha - x \sin \alpha.$$

Hence the product of inertia of the area with respect to these axes is

$$\int x'y'\,dA = \int (x \cos \alpha + y \sin \alpha)(y \cos \alpha - x \sin \alpha) dA$$
$$= \cos 2\,\alpha \int xydA + \tfrac{1}{2} \sin 2\,\alpha (I_X - I_Y). \quad \cdot \quad \cdot \quad (a)$$

We may always determine α so that the product of inertia of the body with respect to the axes OX' and OY' is zero. For if we put $\int x'y'dA = 0$ in equation (a) and solve for α, we get

$$\tan 2\,\alpha = \frac{2\int xydA}{I_y - I_x}, \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (b)$$

which gives a real value for α, for all values of $\int xydA$, I_x and I_y.

237. Principal Axes. — The special case discussed in the preceding article is of great importance. *Any two axes* at right angles to each other, *lying in the plane of the area, for which the product of inertia of the area vanishes are called prin-*

cipal axes of that area, and equation (*b*) shows that *for any point O there are two principal axes.*

Moreover, if we choose the principal axes as the x- and y-axes, respectively, equation (89) shows that the moment of inertia, I_l, about any line through their intersection, making an angle α, with the x-axis is

$$I_l = I_x \cos^2 \alpha + I_y \sin^2 \alpha. \qquad (90)$$

Equation (90) may be written

$$I_l = I_x + (I_y - I_x) \sin^2\alpha. \qquad (90')$$

If now $I_y > I_x$ equation (90′) shows that I_l is the maximum when $\alpha = 90°$, and is then I_y, and a minimum if $\alpha = 0$ and is then I_x, *i.e.* the moment of inertia of the body is a maximum when taken with respect to one of the principal axes, and a minimum when taken with respect to the other. The moment of inertia with respect to a principal axis is called a *principal moment of inertia.* Equation (90′) shows that if $I_x = I_y$, the moment of inertia of the area is the same for all axes through the intersection of the x- and y-axes and in their plane.

It follows from Art. 221 and the present discussion that the least moment of inertia of an area about any axis in its plane is about a principal axis passing through the center of gravity.

238. Least Moment of Inertia. — Engineers need to know the least radius of gyration, and consequently the least moment of inertia of the cross section of every column which they design, since the resistance to flexure is least about that axis which has the least moment of inertia. It was shown in Art. 237, that the moment of inertia of an area is least about a principal axis through the center of gravity. It becomes necessary therefore to determine those principal axes which pass through the center of gravity.

There are two methods of finding such principal axes: first, by symmetry; secondly, by use of the relation given in equation (*b*) of Art. 236.

All axes of symmetry are principal axes. For, taking the axis of symmetry as the y-axis; then, for any area, dA, with coördinates x_1, y_1, there is another equal area dA' with coördinates $-x_1$, y_1 (Art. 92). Hence the quantity

$$\int xydA = \int dA(x_1y_1 - x_1y_1 + x_2y_2 - x_2y_2 + \cdots) = 0.$$

For example, any diameter of a circle, or circular area, is a principal axis; the major and minor axes of an ellipse are principal axes; also, the gravity axes parallel and perpendicular to the web of the cross section of an I-beam, or channel or T-beam, are principal axes.

When the section is unsymmetrical, the angle that the principal axis makes with the x-axis is given by equation (b) of Art. 236.

$$\tan 2\alpha = \frac{2\int xydA}{I_y - I_x}. \quad \ldots \ldots \ldots \quad (a)$$

In this case, therefore, it becomes necessary to evaluate the integral, $\int xydA$. To illustrate the method, we will find the principal axes and the least moment of inertia of the angle $6'' \times 4'' \times \frac{3}{8}''$ (see figure) for which we found $I_x = 4.9$ in.^{4} and $I_y = 13.5$ in.^{4} * in exercise 2 of Art. 233.

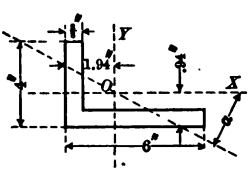


FIG. 132

Let us consider the angle made up of two rectangles A_1, $4'' \times \frac{3}{8}''$, and $A_2 = 5\frac{5}{8}'' \times \frac{3}{8}''$.

To obtain $\int xydA$ for A_1, let $dA = dx \cdot dy$.

Then by double integration with the limits as shown, we have

* See footnote to problem 2, Art. 233.

$$\int xydA = \int_{-1.94}^{-1.565}\int_{-0.94}^{+3.06} xdx\, ydy$$
$$= \int_{-1.94}^{-1.565} xdx\left[\frac{(3.06)^2-(0.94)^2}{2}\right]$$
$$= \left[\frac{(-1.565)^2-(1.94)^2}{2}\right]\left[\frac{(3.06)^2-(0.94)^2}{2}\right]$$
$$= \left(\frac{2.45-3.76}{2}\right)\left(\frac{9.37-0.885}{2}\right) = -2.78 \text{ in.}^4$$

Similarly, $\int xydA$ for the rectangle A_2 is

$$= \int_{-1.565}^{4.06}\quad\int_{-0.94}^{-.565}$$
$$= \int_{-1.565}^{4.06} xdx\left[\frac{(.565)^2-(0.94)^2}{2}\right]$$
$$= \left[\frac{(4.06)^2-(1.565)^2}{2}\right]\left[\frac{(.565)^2-(0.94)^2}{2}\right]$$
$$= -1.98 \text{ in.}^4$$

Therefore $\int xydA$, for the entire section equals

$$\int xydA = -(1.98+2.78) = -4.75 \text{ in.}^4$$

Substituting the values of I_x, I_y and $\int xydA$ in (a), we obtain

$$\tan 2\alpha = \frac{-2\times 4.75}{13.5-4.9} = -\frac{9.5}{8.6} = -1.103;$$
$$\therefore 2\alpha = -47^\circ\ 48';$$
$$\alpha = -23^\circ\ 54'.$$

Now the least moment of inertia is given by equation (89)

$$I = I_x \cos^2\alpha + I_y \sin 2\alpha - \sin 2\alpha \int xydA.$$
$$\therefore I = 2.78 \text{ in.}^4$$

239. Dividing equation (90) by M, the mass of the body, it may be written

$$k_l^2 = k_x^2 \cos^2 \alpha + k_y^2 \sin^2 \alpha, \quad . \quad . \quad . \quad . \quad (91)$$

where k_l, k_x, k_y are respectively the radius of gyration of the body with respect to l, the x-axis and y-axis.

240. The Momental Ellipse. — On any line l, through O, making an angle α with the x-axis, which is one of the principal axes, lay off $OP = r$, where $r = \frac{k_x k_y}{k_l}$ (see Art. 239). Then, as α takes all values, the point P will describe an ellipse called the momental ellipse. For, let the coördinates of P, referred to OX and OY, be x, y.

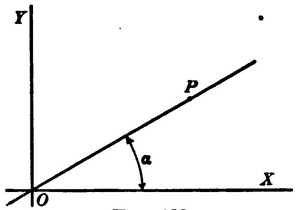


FIG. 133

Then

$$\cos \alpha = \frac{x}{r} = \frac{k_l x}{k_x k_y},$$

$$\sin \alpha = \frac{y}{r} = \frac{k_l y}{k_x k_y}.$$

Substituting in equation (91) we get

$$\frac{x^2}{k_y^2} + \frac{y^2}{k_x^2} = 1.$$

REMARK. — The theorems developed in Arts. 234–240, for plane areas, apply with equal force to thin plane laminæ, if we substitute the word Mass for Area.

241. Products of Inertia of a Solid. — We have in the main dealt with plane figures. The solids we have considered have been built up of plane laminæ. There are analogous definitions and theorems for bodies in three dimensions. For example, the moment of inertia of a body with respect to a *plane* is the sum of the products formed by multiplying each elementary mass by the square of its distance from the plane. If the coordinates of any mass m referred to three planes mutually perpendicular to each other are x, y, z, the quantities Σmxy, Σmyz,

Σmxz are called the products of inertia of the body. If the products of inertia are all zero, the coördinate planes are called principal planes, and the coördinate axes are called principal axes.

It is at once evident that any plane of symmetry is a principal plane. For suppose the yz-plane is a plane of symmetry; then if there is a differential mass, dm, whose coördinates are $+x$, y, z, there is another equal dm whose coördinates are $-x$, y, z.

Hence, $\Sigma xydm = 0$, and $\Sigma xzdm = 0$, for the terms in these expressions cancel each other in pairs. Similarly if either the xz-plane or the xy-plane is a plane of symmetry, $\Sigma yxdm = \Sigma yzdm = 0$, and $\Sigma zxdm = \Sigma yzdm = 0$ respectively. It can be shown that for any point there are three principal planes.

If the principal axes are the axes of coördinates and l any line through the origin whose direction angles are α, β, γ, then it can be shown that

$$I_l = I_x \cos^2 \alpha + I_y \cos^2 \beta + I_z \cos^2 \gamma,$$

and instead of a momental ellipse there is a momental ellipsoid.

242. Exercises

Find the moment of inertia:

1. Of a rectangle, base b and altitude a, with reference to a diagonal. *Ans.* $I = \frac{1}{6}\frac{a^3b^3}{a^2+b^2}$.

2. Of an ellipse about a diameter making an angle α with the major axis.

3. Find the momental ellipse of a square.

4. Find the momental ellipse of a hexagon.

CHAPTER XV

THE DYNAMICS OF A RIGID BODY

243. We shall regard a rigid body as a continuous collection of material particles so connected that the distance between any two of them is constant. We shall consider first the motion of the rigid body, without considering the cause of the motion.

244. Displacement of a Rigid Body. Plane Motion. — Because the problem is so much simpler, not only from the standpoint of the development of the theorems, but also from that of their application to actual problems, and yet general enough to include most problems with which we have to deal, we shall for the present confine ourselves to the consideration of the motion of bodies the particles of which move parallel to a fixed plane; as, for example, a book sliding on a table, a train moving on a straight track, machinery moving in a straight groove, a cylinder rolling on a plane, a flywheel rotating round an axis, etc. If we think of the body as made up of plane laminæ parallel to the fixed plane, the displacement of any one of these plane laminæ is exactly the same as that of any other; hence we need consider the displacement of one of these laminæ only. We shall speak of such motion as *plane motion*, or *motion of a body in a plane.*

Since a plane area is fixed in its plane if two of its points are fixed, the displacement of any lamina is completely known if the displacement of any two of its points is known. Hence in plane motion we need consider the displacement of two points only. Now, a rigid body can have a motion either of translation, of rotation, or of translation and rotation.

245. Translation. — If a body move so that any given line in it remains parallel to itself, the motion is called translation. It follows at once that the displacements of all particles in the body are represented by equal and parallel straight lines. For suppose that in the translation of the body M, the line AB fixed in the body takes the position $A'B'$. Complete the parallelogram $ABB'A'$. Then since M is a rigid body it follows that $AA' = BB'$.

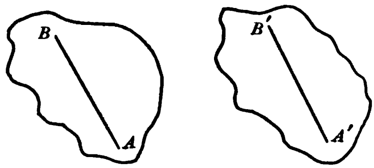


FIG. 134

It further follows that at any instant the velocity (and also the acceleration) of any point A is the same in magnitude and direction as the velocity (or the acceleration) of any other point B. And hence if the body is translating, the motion of a single point completely defines the motion of the entire body; thus the problem of the motion of a rigid body in translation is reduced to that of the motion of a particle.

246. Rotation. — If a body turns round a fixed axis the motion is one of rotation. Each particle not on the axis of rotation describes a circle whose center is on the axis and whose plane is perpendicular to the axis. An example is the flywheel. Consider a plane lamina, rotating around an axis perpendicular to its plane, piercing the plane in O. By this rotation a line AB, fixed in the lamina, is displaced to $A'B'$. Now since the body is rigid,

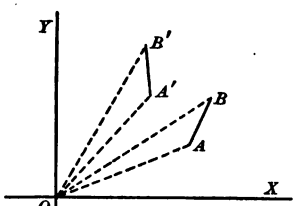


FIG. 135

$$\angle AOB = \angle A'OB'.$$

Adding algebraically to both sides of this equation, the angle BOA' we obtain
$$\angle BOB' = \angle AOA';$$
that is, the angular displacement of B equals the angular displacement of A.

Therefore the ***angular velocity***, ω, and ***angular acceleration***, α, of any point A equal respectively the angular velocity and angular acceleration of any other point B. Hence, the angular motion is completely determined by that of any single point not on the axis. The linear velocity of a point in a rotating body = the distance of the point from the axis of rotation multiplied by the angular velocity of the body (Art. 168).

247. Theorem. — *Any displacement of a rigid body parallel to a fixed plane may be made by a single rotation.* For suppose AB be displaced to $A'B'$. Erect a perpendicular at the mid-point of AA' and BB' respectively. Let their intersection be C. Then

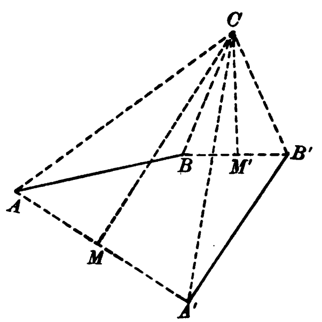


FIG. 136

$$CA = CA', \text{ and } CB = CB'.$$

And from triangles ACB and $A'CB'$,

$$\angle ACB = \angle A'CB';$$

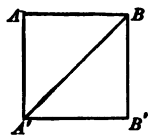
therefore, $\angle ACA' = \angle BCB'$;

whence it follows that a rotation about C will displace AB to $A'B'$.

The student will note that this construction will not determine C if AA' is parallel to BB'. In this case it is evident that C is at the intersection of AB and $A'B'$ produced (if necessary). Moreover, if AB is parallel to $A'B'$, C is at infinity. That is, a translation may be regarded as a rotation about an infinitely distant axis. If the displacement AA' and BB' approaches zero, the limiting position of the point C is called the instantaneous center. It is possible to show that any displacement of a plane may be made by rolling a curve fixed in the moving lamina over a curve fixed in space. (See Ziwet's *Theoretical Mechanics*, Part I, pp. 8 *et seq.*)

The student may prove the corollary: Any displacement of a body parallel to a plane may be made by a rotation and a translation, or a translation and a rotation.

248. **Exercises**



1. Find C and the angle through which the square $ABB'A'$ must be rotated so that the side AB will take the direction $A'B$ and, after displacement, A will occupy the position that A' did before displacement.

2. Show that if a plane area is rotated through an angle θ about an axis through any point O, and then rotated through an angle $-\theta$ about a parallel axis through any other point O', the resultant displacement is a translation.

249. External and Internal Forces. — We shall now consider the behavior of a *rigid body* under the action of forces. To derive the equations of the mechanics of a rigid body, let us suppose that forces $F_1, F_2 \cdots F_n$ be applied to the body at the points P_1, $P_2 \cdots P_n$, respectively, whose coördinates are respectively x_1, y_1, z_1; $x_2, y_2, z_2 \cdots x_n, y_n, z_n$. Let the resolved parts of $F_1, F_2 \cdots$ along the axes of coördinates be X_1, Y_1, Z_1; $X_2, Y_2, Z_2 \cdots$ respectively. Let the weights of the particles at P_1, $P_2 \cdots$ be $W_1, W_2 \cdots$ respectively. Let us consider any point P_i at which a force F_i acts on a particle whose weight is W_i. Then there are two types of forces acting on P_i. (1) Those of the nature of F_i which are applied to it and are called *external* or *applied* forces. (2) Those groups of forces which are due to the interaction of the particles, such as forces of cohesion, which prevents the separation of the particles, or of pressure of the particles on each other, which resists compression; in short, all forces which tend to prevent the deformation of the body. These forces are called *internal* forces. For example, in considering the stresses in trusses as in Art. 85, the external forces were the loads applied at the joints and

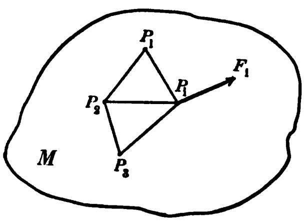


FIG. 137

the reactions at the piers; the effects of these external forces were communicated by the rigid members of the truss to pins at other points than those at which the forces were applied, and these stresses were in turn communicated to members to which loads were not directly applied. The stresses in the members of the truss are the internal forces.

If a body slides down a rough inclined plane, the external forces that are applied to the body are its weight and the reaction of the plane. If one considers the various particles of which the body is composed, each particle is acted upon by its weight and also by some other force whether the particles are in contact with the plane or not, otherwise they would describe vertical lines. Certain effects of the reaction of the plane on those particles in contact with it are communicated to other particles; these effects are the internal forces.

250. A Rigid Body under the Action of Force. — Each particle of the rigid body may be acted upon by any number of external forces, which we may replace by a resultant F, and any number of internal forces, which we may replace by a resultant F'.

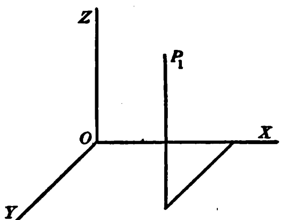


FIG. 138

Let the body consist of n particles referred to the mutually perpendicular axes OX, OY, OZ, *fixed in space*. Let the particle at P_1, whose coördinates are x_1, y_1, z_1, and whose weight is W_1, be acted upon by an external force F_1, whose components parallel to the axes are X_1, Y_1, Z_1, and by an internal force F'_1, whose components parallel to the axes are X'_1 Y'_1 Z'_1. Let the acceleration of P parallel to the x-, y-, z-axes be a_x, a_y, a_z, respectively. Similarly let the particle at P_2, whose coördinates are x_2, y_2, z_2, and whose weight is W_2, be acted upon by an external force F_2, and an internal force F'_2, whose components are respectively X_2, Y_2, Z_2 and X'_2, Y'_2, Z'_2, and similarly

for all the other particles. Then the equations of motion of the particles are

$$\left.\begin{array}{l}
\frac{W_1}{g}a_{x_1} = X_1 + X'_1;\ \frac{W_1}{g}a_{y_1} = Y_1 + Y'_1;\ \frac{W_1}{g}a_{z_1} = Z_1 + Z'_1. \\
\frac{W_2}{g}a_{x_2} = X_2 + X'_2;\ \frac{W_2}{g}a_{y_2} = Y_2 + Y'_2;\ \frac{W_2}{g}a_{z_2} = Z_2 + Z'_2. \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ;\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ;\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ;\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ;\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\frac{W_i}{g}a_{x_i} = X_i + X'_i;\ \frac{W_i}{g}a_{y_i} = Y_i + Y'_i;\ \frac{W_i}{g}a_{z_i} = Z_i + Z'_i. \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ;\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ;\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}\right\} \quad (92)$$

Adding the first column of the equations (92), then the second column, then the third column, we get

$$\left.\begin{array}{l}
\frac{W_1}{g}a_{x_1} + \frac{W_2}{g}a_{x_2} + \cdots = X_1 + X_2 + \cdots X'_1 + X'_2 + \cdots; \\
\frac{W_1}{g}a_{y_1} + \frac{W_2}{g}a_{y_2} + \cdots = Y_1 + Y_2 + \cdots Y'_1 + Y'_2 + \cdots; \\
\frac{W_1}{g}a_{z_1} + \frac{W_2}{g}a_{z_2} + \cdots = Z_1 + Z_2 + \cdots Z'_1 + Z'_2 + \cdots;
\end{array}\right\} \quad . \quad . \quad (a)$$

or

$$\left.\begin{array}{l}
\sum \frac{W_i}{g}a_{x_i} = \Sigma X_i + \Sigma X'_i; \\
\sum \frac{W_i}{g}a_{y_i} = \Sigma Y_i + \Sigma Y'_i; \\
\sum \frac{W_i}{g}a_{z_i} = \Sigma Z_i + \Sigma Z'_i.
\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (a')$$

Let us now take moments in order about the z-, x-, and y-axes. The moment M_i of the forces acting at P_i, about the z-axis is, Art. 67,*

$$M_i = Y_i x_i - X_i y_i + Y'_i x_i - X'_i y_i.$$

If we put $\Sigma M_i = M_z$,

$$M_z = \Sigma(Y_i x_i - X_i y_i) + \Sigma(Y'_i x_i - X'_i y_i).$$

* Since the convention regarding signs of moment is immaterial, we have considered clockwise moments as positive in this article, instead of negative as in Art. 67.

If we multiply the ith equation of column 2 in equations (92) by x_i, and the ith equation of column 1 by y_i, and subtract the latter from the former, and sum for all the particles of the body, we get

$$\left.\begin{aligned}
M_z &= \Sigma(Y_i x_i - X_i y_i) + \Sigma(Y'_i x_i - X'_i y_i) \\
&= \sum \frac{W_i}{g}(a_{y_i} x_i - a_{x_i} y_i);
\end{aligned}\right.$$

Similarly, if M_x and M_y are the moments of the forces about the x- and y-axes respectively,

$$\left.\begin{aligned}
M_x &= \Sigma(Z_i y_i - Y_i z_i) + \Sigma(Z'_i y_i - Y'_i z_i) \\
&= \sum \frac{W_i}{g}(a_{z_i} y_i - a_{y_i} z_i); \\
M_y &= \Sigma(X_i z_i - Z_i x_i) + \Sigma(X'_i z_i - Z'_i x_i) \\
&= \sum \frac{W_i}{g}(a_{x_i} z_i - a_{z_i} x_i).
\end{aligned}\right\} \quad . \; . \quad (b)$$

We shall now assume that the *internal forces* are *in equilibrium among themselves*. Hence, Art. 79,

$$\left.\begin{aligned}
&\Sigma X' = \Sigma Y' = \Sigma Z' = 0; \\
\text{and} \quad &\Sigma(Y'_i x_i - X'_i y_i) = \Sigma(Z'_i y_i - Y'_i z_i) = \Sigma(X'_i z_i - Z'_i x_i) = 0.
\end{aligned}\right\} \quad (c)$$

Substituting the values from (c) in equations (a) and (b) we obtain

$$\left.\begin{aligned}
\sum \frac{W_i}{g} a_{x_i} &= \Sigma X_i = X, \text{ (say)} \\
\sum \frac{W_i}{g} a_{y_i} &= \Sigma Y_i = Y, \text{ (say)} \\
\sum \frac{W_i}{g} a_{z_i} &= \Sigma Z_i = Z. \text{ (say)}
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (93)$$

$$\left.\begin{aligned}
M_z &= \Sigma(Y_i x_i - X_i y_i) = \sum \frac{W_i}{g}(a_{y_i} x_i - a_{x_i} y_i); \\
M_x &= \Sigma(Z_i y_i - Y_i z_i) = \sum \frac{W_i}{g}(a_{z_i} y_i - a_{y_i} z_i); \\
M_y &= \Sigma(X_i z_i - Z_i x_i) = \sum \frac{W_i}{g}(a_{x_i} z_i - a_{z_i} x_i).
\end{aligned}\right\} \quad . \; . \; . \quad (94)$$

251. Equations (93) and (94) are called the equations of motion of a rigid body. It will be observed that there are six equations defining the behavior of a rigid body under the action of forces, three of which pertain to the acceleration of the body along axes mutually at right angles to each other, and three of which pertain to the rotation of the body about these axes. The student will recall that in the statics of a rigid body he found six equations of equilibrium. Three of these equations were obtained by equating to zero the sum of the resolved parts of the forces along the axes, and three were obtained by equating to zero the moments of the forces about these axes. He will note that these equations of statics are special cases of the ones just developed in which he has (since the acceleration is zero) written the right-hand side of the equation equal to zero.

Suppose now that the right-hand side of one of the equations of Art. 250 were not zero, say, for example, $\Sigma X_i = X$ (some force), but that the remaining five equations were zero. Then the body would move parallel to the x-axis. Similarly, if the right-hand side of any number of the equations (93) and (94) were not zero, it is evident that the body would move as prescribed by these forces. These are called constraints, and in all, one can give to any rigid body six and only six constraints; and if no constraints are given, the body can move in six ways. We usually phrase this by saying that a body may have six degrees of freedom, and in general that the degree of freedom of a body is six minus the number of constraints.

252. The Motion of the Center of Gravity. — We shall now consider some special but very important cases, considering first the motion of the center of gravity.

If $\bar{x}$, $\bar{y}$, $\bar{z}$ be the coördinates of the center of gravity of the body, and $W = W_1 + W_2 + \cdots$, then (Art. 93)

$$\left.\begin{aligned} W\bar{x} &= W_1x_1 + W_2x_2 + \cdots, \\ W\bar{y} &= W_1y_1 + W_2y_2 + \cdots, \\ W\bar{z} &= W_1z_1 + W_2z_2 + \cdots. \end{aligned}\right\} \quad \cdots\cdots \quad (a)$$

Differentiating each of equations (a) with respect to t twice and substituting in equations (93), we obtain the very important relation

$$\left.\begin{aligned}\frac{W}{g}\frac{d^2\bar{x}}{dt^2} &= X,\\ \frac{W}{g}\frac{d^2\bar{y}}{dt^2} &= Y,\\ \frac{W}{g}\frac{d^2\bar{z}}{dt^2} &= Z.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

That is, the *center of gravity of a rigid body moves as if the entire mass of the body were concentrated at that point and acted upon by forces (applied there) equal and parallel to the external forces.*

253. Let us now consider the special case in which the resultant of all the external forces acting on the body is zero.

Then

$$\frac{d^2\bar{x}}{dt^2} = 0, \quad \frac{d^2\bar{y}}{dt^2} = 0, \text{ and } \frac{d^2\bar{z}}{dt^2} = 0\,;$$

hence

$$\frac{d\bar{x}}{dt} = c, \quad \frac{d\bar{y}}{dt} = c', \text{ and } \frac{d\bar{z}}{dt} = c'',$$

where c, c', and c'' are constants of integration.

Hence, if $\bar{v} =$ velocity of the center of gravity,

$$\bar{v}^2 = \left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2 + \left(\frac{d\bar{z}}{dt}\right)^2 = c^2 + c'^2 + c''^2 = \text{a constant.}$$

Another integration gives

$$\begin{aligned}\bar{x} &= ct + d,\\ \bar{y} &= c't + d',\\ \bar{z} &= c''t + d'',\end{aligned}$$

where d, d', and d'' are constants of integration.

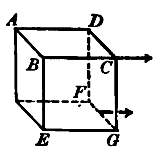
Therefore $\dfrac{\bar{x} - d}{c} = \dfrac{\bar{y} - d'}{c'} = \dfrac{\bar{z} - d''}{c''}$, which is the equation of a straight line.

That is, if the resultant of all the external forces applied to a body is zero, the center of gravity will move in a straight line with uniform velocity. This is a generalization of Newton's First Law of Motion.

REMARK. — It will be noticed that the above reasoning applies also to a non-rigid system of particles if the internal forces are in equilibrium among themselves. For example, so far as astronomers can tell, the resultant of all the external forces acting on the bodies of the solar system is zero; and since the internal forces, that is, the mutual attraction of the bodies for each other, are in equilibrium, the center of gravity of the solar system is moving in a straight line with uniform velocity.

254. Translation. — By Art. 245, all points in a translating body have at any instant the same direction, velocity, and acceleration, and are describing parallel curves. The case of most frequent occurrence and hence of the greatest importance is that in which each particle, and therefore the center of gravity, describes a straight line. Now, a particle will describe a straight line if, and only if, it is acted on by a force in the line of motion of the particle. Therefore the *motion of a body is one of translation if, and only if, the resultant of all the applied forces acting on it passes through the center of gravity of the body and acts in the direction in which the body is moving.* It therefore follows that in translation the sum of the moments of the external forces about any axis through the center of gravity is zero.

255. Exercises



1. A cube $ABCDEFG$, one of whose edges $AB = 2$ feet, rests on a smooth horizontal plane. A force of 5 pounds acts at C perpendicular to the face DCF. A parallel force of 2 pounds acts on the middle point of FG. Find a parallel force R applied to the edge DF so that the cube will translate. Find the point of application of R.

Ans. $R = 5$ pounds applied $\frac{2}{5}$ feet above F.

2. In the preceding problem, suppose the force of 2 pounds acts in the opposite direction. Find R and its point of application.

3. In example 1 find the acceleration of the cube at the end of 5 seconds, assuming that the cube weighs 100 pounds.

256. Rotation of a Body about a Fixed Axis. — Let us suppose a body is rotating about an axis fixed at two points A and B.

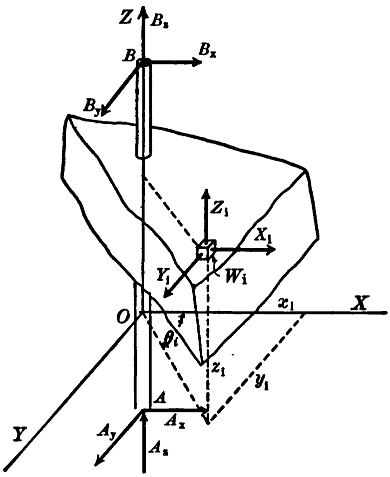


FIG. 139

Choose the axis of rotation as the z-axis. Let the components of the applied forces at the particle P_i, whose weight is W_i, be X_i, Y_i, Z_i, and the coördinates of P_i be x_i, y_i, z_i. Let the components of the reactions on the axes at A and B be A_x, A_y, A_z, and B_x, B_y, B_z. Let the weight of the entire body be W. If a line from P_i perpendicular to the z-axis makes the angle θ with the xz-plane,

$$x_i = r_i \cos \theta_i;$$
$$y_i = r_i \sin \theta_i;$$

whence

$$\frac{dx_i}{dt} = -r_i \sin \theta_i \frac{d\theta}{dt} = -y_i\omega; \quad \frac{dy_i}{dt} = x_i\omega;$$

$$a_{x_i} = \frac{d^2x_i}{dt^2} = -\frac{dy_i}{dt}\omega - y_i\frac{d\omega}{dt} = -x_i\omega^2 - y_i\alpha; \quad a_{y_i} = \frac{d^2y_i}{dt^2} = -y_i\omega^2 + x_i\alpha,$$

where ω and α are respectively the angular velocity and angular acceleration of the body (Art. 168). Since the rotation is parallel to the xy-plane, $a_{z_i} \equiv 0$, and equations (93) and (94) may in this case be written:

$$\left.\begin{aligned}
\sum \frac{W_i}{g} a_{x_i} &= \Sigma X_i + A_x + B_x = -\frac{W}{g}\omega^2\bar{x} - \frac{W}{g}\alpha\bar{y}.\\
\sum \frac{W_i}{g} a_{y_i} &= \Sigma Y_i + A_y + B_y = -\frac{W}{g}\omega^2\bar{y} + \frac{W}{g}\alpha\bar{x}.\\
0 &= \Sigma Z + A_z + B_z.\\
M_z &= \sum \frac{W_i}{g}[(-y_i\omega^2 + x_i\alpha)x_i - (-x_i\omega^2 - y_i\alpha)y_i]\\
&= \alpha \sum \frac{W_i}{g}(x_i^2 + y_i^2)\\
&= I\alpha;\\
M_x &= \sum \frac{W_i}{g}(y_i\omega^2 - x_i\alpha)z_i\\
&= \omega^2 \sum \frac{W_i}{g} z_i y_i - \alpha \sum \frac{W_i}{g} x_i z_i;\\
M_y &= \sum \frac{W_i}{g}(-x_i\omega^2 - y_i\alpha)z_i\\
&= -\omega^2 \sum \frac{W_i}{g} z_i x_i - \alpha \sum \frac{W_i}{g} y_i z_i.
\end{aligned}\right\} \quad . \quad (a)$$

If now we make the body "free" by inserting not only the forces acting at the various points of the body, but the reactions on the axis as well, and let the ΣX, ΣY, and ΣZ be the resolved parts of *all forces acting on the body*, equation (a) may be written as follows (we assume that the body is continuous and hence use the integral instead of the summation signs):

$$\left.\begin{aligned}
\Sigma X &= \frac{-W}{g}\omega^2\bar{x} - \frac{W}{g}\alpha\bar{y},\\
\Sigma Y &= \frac{-W}{g}\omega^2\bar{y} + \frac{W}{g}\alpha\bar{x},\\
Z &= 0.\\
M_z &= \text{moment of external forces about } z\text{, the axis of rotation}\\
&= I\alpha.\\
M_x &= \omega^2 \int zy\frac{dW}{g} - \alpha \int zx\frac{dW}{g}.\\
M_y &= -\omega^2 \int zx\frac{dW}{g} - \alpha \int yz\frac{dW}{g}.
\end{aligned}\right\} \quad . \quad (96)$$

257. Equation of Angular Motion. — Let us consider now the fourth of equations (96)

$$M_z = I\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

It will be recalled that the z-axis is the axis of rotation and M_z the *moments of the applied* forces about this axis. Equation (97) translated into words is

The moment of the external forces about any fixed axis of rotation l, equals the moment of inertia of the body about that axis multiplied by the angular acceleration of the body.

Equation (97) is sometimes called the equation of angular motion of the body and is an important one.

258. The Compound Pendulum. — To illustrate the use of equation (97), let us consider the compound pendulum.

A rigid body suspended from a horizontal axis l, about which it may rotate under the action of its own weight, is called a compound pendulum.

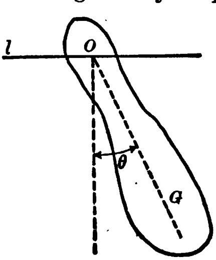


FIG. 140

Problem. — To determine the motion of a compound pendulum.

The external forces acting on the pendulum are the reactions on the axis, the moment of which about l is zero, and the weight of the pendulum which may be regarded as acting at G, its center of gravity (Art. 89). Pass a plane through G, perpendicular to the axis l and intersecting it at O. Let

W = weight of the body;

k_l = its radius of gyration about the axis l;

$OG = L$.

I_l = moment of inertia of the body about l;

θ = angle that OG makes with the vertical.

Since the moment of the external forces about l is $WL \sin \theta$, equation (97) becomes

$$W \cdot L \sin\theta = -I_l \frac{d^2\theta}{dt^2}$$

or

$$\frac{d^2\theta}{dt^2} = \frac{-WL \sin\theta}{\frac{W}{g}k_l^2} = -\frac{-gL\sin\theta}{k_l^2}. \quad . \quad . \quad . \quad . \qquad (a)$$

If we put $l = \frac{k_l^2}{L}$, we see that equation (a) is the same as equation (71) of Art. 196. Hence a compound pendulum will rotate about an axis in exactly the same way as a simple pendulum whose length is

$$l = \frac{k_l^2}{L}.$$

For example, the time of a small vibration is, Art. 196,

$$t = \pi\sqrt{\frac{k_l^2}{gL}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

The length, $l = \frac{k_l^2}{L}$, is called the *length of the equivalent simple pendulum.*

For example, let a compound pendulum consist of a right circular cylinder, joined by a weightless rod coincident with the axis of the cylinder, and intersecting a horizontal axis at right angles. Let $c =$ the length of the rod OA (*i.e.* the distance of the axis from nearest base);

$r =$ radius of base of the cylinder;

$2h =$ altitude of the cylinder.

Then the time of a small vibration of such a pendulum is

$$t = \pi\sqrt{\frac{k_l^2}{(c+h)g}}.$$

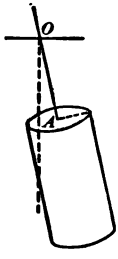


FIG. 141

259. From Art. 221 $k_l^2 = k^2 + L^2$, where k is the radius of gyration of the body about a gravity axis parallel to the fixed axis, and is, for a given body, a constant.

Equation (*b*), Art. 258, may now be written

$$t = \pi\sqrt{\frac{k^2 + L^2}{Lg}}. \quad \ldots \ldots \quad (98)$$

From this equation it is evident that there exists some value of L such that t may be a minimum. To find this, differentiate t with regard to L, and put $\frac{dt}{dL}$ equal to zero. There results $L = k$, which is a minimum or a maximum. It cannot be a maximum for it requires an infinite time for the body to rotate about an axis through the center of gravity. Hence the pendulum will vibrate in the shortest time if its axis pass through a point O such that $L = k$; and the time of a small vibration is

$$t = \pi\sqrt{\frac{2k}{g}}.$$

In order therefore that the time of vibration of a plane lamina about an axis in its plane shall be the shortest possible, the fixed axis must be parallel to that one of the principal axes through the center of gravity about which the moment of inertia is least, and at a distance k from this axis.

260. Center of Suspension. Center of Oscillation. — The point O is called the *center of suspension*. The point C taken on the line OG, such that

$$OC = a = \frac{k^2 + L^2}{L}$$

is called the *center of oscillation*. It is evident that $a > L$, and hence C and O lie on opposite sides of G. Also if

$$L' = GC,$$
$$L' = a - L.$$

If now the pendulum is suspended about a parallel axis through C, the time of a small vibration is

$$t = \pi\sqrt{\frac{L'^2 + k^2}{gL'}} = \pi\sqrt{\frac{L^2 + k^2}{gL}},$$

which the student may verify.

That is, the time of a small vibration is the same about an axis through the center of suspension, as that of a small vibration about a parallel axis through the center of oscillation.

261. We shall derive equation (97) by another method — a method which for many problems is more direct than that of Art. 256. Consider as usual the body as made up of a great number of particles and consider any particle P_i acted upon by the force F_i, the resultant of *all* the forces acting on the particle. Let F_i be resolved parallel and perpendicular to $O'P_i$, the component perpendicular to the line of $O'P_i$ being called dT_i and the component along $O'P_i$ being called dN_i. It is evident that these components are respectively tangent and normal to the path of the particle at that instant. Let dW_i be the weight of the particle P_i. If α be the angular acceleration of P_i, the linear acceleration $= \alpha r_i$ in which $r_i = O'P_i$ the distance of the particle from the axis.

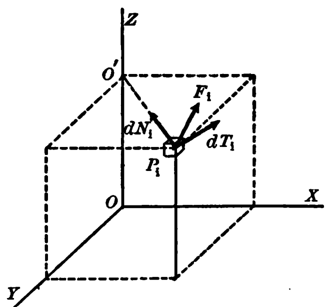


FIG. 142

Therefore $dT_i = \frac{dW_i}{g}\alpha r_i$ (Second Law of Motion).

Now let us multiply each side of the equation by r_i, and sum the series of equations thus found.

Then

$$\int r_i dT_i = \int \alpha r_i^2 \cdot \frac{dW_i}{g}$$

But $\int r_i dT_i$ is evidently the sum of the moments about the axis OZ of the resultant force acting upon each particle, and since the sum of the moments of the internal forces in any rigid body about any axis is zero, $\int r_i dT_i$ must equal the mo-

ments about the axis OZ of the external forces acting upon the body.

Again, since α is the same for all particles, the right side of the equation may be written

$$\alpha\int r_i^2\frac{dW_i}{g}.$$

But

$$\int r_i^2\frac{dW_i}{g}=I.$$

Hence the sum of the moments of the external forces acting upon a body rotating about a fixed axis $= I\alpha$.

262. Exercises

1. A circular disc of radius 4 inches swings around a horizontal tangent. Find the length of the equivalent simple pendulum. In what time will the pendulum vibrate?

Ans. 5 inches; .357 seconds.

2. A circular disc revolves about an axis perpendicular to its plane. Find a position of the axis in order that it may vibrate in the shortest time.

3. Find the length of the simple pendulum that will vibrate in the same time as a right circular cone about an axis through the vertex parallel to the base of the cone.

Ans. $\dfrac{4h^2+r^2}{5h}$, where h is the height of the cone and r the radius of the base.

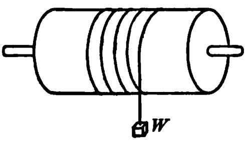
4. Find the center of oscillation of the cone in example 3.

5. Find a center of suspension such that a rectangle 2 feet by 4 feet may vibrate once in 2 seconds about an axis perpendicular to its plane.

Ans. Anywhere on the circumference of two circles, whose centers are at the center of the rectangle and whose radii are respectively 12.9 feet and .13 feet.

6. If the cone in exercise 3 start from rest when the axis of the cone is inclined 45° to the vertical, find its angular velocity when the axis is vertical.

7. Cast iron weighs 450 pounds per cubic foot. A cast-iron right circular cylinder with radius of base 1 foot and height 1 foot is supported with its axis horizontal by two axles at the center of the bases. The cylinder may turn round its axis without friction. A weight of 20 pounds is suspended by a string which is fastened to the cylinder after being wrapped round its circumference many times.



Suppose the axles of the cylinder are weightless. Find the angular acceleration of the cylinder, the acceleration of the weight, and the tension of the string. Find how far the weight will descend in ten seconds supposing it start from rest.

Ans. Ang. acc. $= \dfrac{20\,g}{225\,\pi + 20}$ radians per second per second.

Acc. of $W = \dfrac{20\,g}{225\,\pi + 20}$ feet per second per second.

Tension $= \dfrac{4500}{225\,\pi + 20}$ pounds.

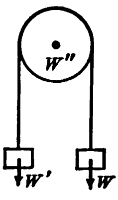
8. Let the cylinder in exercise 7 have a radius of 9 inches instead of 1 foot and let the weight slide down a frictionless plane which makes an angle of 45° with the horizontal. Let all other data remain as in exercise 7. Find the angular acceleration of the cylinder, the linear acceleration of the weight, the tension of the string, and the angle which the cylinder turns through in 6 seconds.

9. Find a center of suspension such that a rectangular lamina 3 by 5 inches will vibrate in the shortest possible time about an axis,

(*a*) in the plane of the lamina;

(*b*) perpendicular to the plane of the lamina.

10. Two weights, W and W', $W > W'$, are connected by a weightless cord, which passes without slipping over a homogeneous cast-iron pulley in the form of a cylinder, of radius r, and height h, and which rotates without friction about an axis perpendicular to the plane of its base. Find the acceleration of W and W', and the tension in the cord to which W and W' are attached. Assume the weight of the pulley to be W''.



11. The pendulum of a clock consists of a brass bob in the form of a flat thin cylindrical disc 2 inches in diameter and $\frac{1}{4}$ inch thick attached to a thin cylindrical steel rod, $\frac{1}{8}$ inch radius, the rod extending to the top of the disc. The mechanism vibrates in the plane of the disc. How long should the rod be in order that the time of a small oscillation may be one second? Brass weighs 525 pounds per cubic foot. Steel weighs 490 pounds per cubic foot.

12. By how much should the length of the rod be changed in order that the pendulum may lose 2 seconds each 24 hours?

13. How long should the rod be if the pendulum, in example 11, oscillates once a second when it swings through an angle of 60°?

Hint. — See Art. 198.

263. The Pressure on the Axis of Rotation. — We may by solving equations (96) obtain in terms of the given forces and other known quantities A_x, A_y, B_x, B_y, ω, and α, since $\alpha = \frac{d\omega}{dt}$. These quantities give the reaction on the axis perpendicular to it, but leave undetermined the reactions A_z and B_z. Since, however, these tend to produce motion of the axis of rotation parallel to itself, and the axis is usually so fixed that this motion is impossible, we seldom are required to find these reactions.

Now each of the left-hand side of the first two equations of

(*a*), Art. 256 represents a single force, and hence the resultant of the two forces is a single force or a couple. We can show that it is a single force by the last two of equations (*a*) of Art. 256. Remembering that the coördinates of A and B are respectively 0, 0, a and 0, 0, b, these equations may be written

$$M_x = A_y a - B_y b = \omega^2 \sum \frac{W}{g} zy - \alpha \sum \frac{W}{g} xz.$$

$$M_y = B_x b - A_x a = -\omega^2 \sum \frac{W}{g} zx - \alpha \sum \frac{W}{g} yz.$$

If, therefore, the xy-plane is a plane of symmetry, $\sum \frac{W}{g} zx = \sum \frac{W}{g} zy = 0$. Hence $M_x = M_y = 0$, and hence there is no tendency of the body to rotate about either the x- or the y-axis, and therefore the left-hand side of first two equations of (*a*), Art. 256, can be replaced by a single force R, making an angle θ with the x-axis.

If, further, $\alpha = 0$, R passes through the center of gravity; for,

$$R \cos \theta = \Sigma X + A_x + B_x = -\omega^2 \frac{W}{g} \bar{x};$$

$$R \sin \theta = \Sigma Y + A_y + B_y = -\omega^2 \frac{W}{g} \bar{y}.$$

Hence $\tan \theta = \frac{\bar{y}}{\bar{x}}$; and R, therefore, passes through the center of gravity.

Moreover, $R = \frac{W}{g} \omega^2 \sqrt{\bar{x}^2 + \bar{y}^2} = \frac{W}{g} \omega^2 \bar{r}.$

R therefore equals the centrifugal force of a particle whose weight is W situated at the center of gravity of the body, and rotating with constant angular velocity ω.

It will be noted that $\sum \frac{W}{g} xz = 0 = \sum \frac{W}{g} yz$ are the conditions that the z-axis be a principal axis. It will be noted further that the xy-plane contains the center of gravity, since it is a plane of symmetry.

If the axis pass through the center of gravity and hence $\bar{x} = 0 = \bar{y}$, it follows that $R = 0$. If these conditions are not satisfied, there will be a pressure on the axis, which is constantly changing its direction, and hence causing a rapidly rotating mechanism to vibrate.

264. **Exercises**

A right circular cone, whose radius is 1 foot, and altitude 2 feet, and weight 100 pounds, may rotate about a horizontal axis through the vertex parallel to the base. The cone is displaced so that its axis is horizontal, and then allowed to vibrate.

1. Find its angular velocity ω when the axis of the cone is vertical. *Ans.* $\omega^2 = \frac{20}{17} g$.

2. Find its angular velocity when the axis of the cone makes an angle of 45° with the vertical.

3. Find the total pressure on the axis of rotation in exercise 1. *Ans.* 276 pounds (nearly).

4. Find in exercise 2 the pressure on the axis due to rotation.

265. Kinetic Energy. — If a body is composed of n particles weighing respectively $W_1, W_2, \cdots$ having respectively velocities of $v_1, v_2, \cdots$, the *kinetic energy*, E, of the body is defined by the equation

$$E = \frac{1}{2} \sum \frac{W_i}{g} v_i^2. \qquad (99)$$

There are three cases of special interest:

(1) *Translation.* In this case the velocity of all the particles is the same, hence v may be taken from under the sign of summation. Hence,

$$E = \frac{1}{2} v^2 \sum \frac{W_i}{g} = \frac{1}{2} \frac{Wv^2}{g}, \qquad (100)$$

where $W = \Sigma W_i$.

That is, the kinetic energy of the body is the same as that of a particle (Art. 186) whose mass equals that of the body.

(2) *Rotation about a fixed axis.* If ω is the angular velocity of the body, and v_i and r_i are respectively the velocity of W_i and its distance from the axis of rotation, then

$$v_i = r_i\omega.$$

Hence the equation (99) becomes

$$E = \frac{1}{2}\omega^2\sum\frac{W_i}{g}r_i^2 = \frac{1}{2}I_l\omega^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

where I_l = moment of inertia of the body about the fixed axis l.

(3) *Motion in a plane. Combined rotation and translation.* Consider a body M, whose particles are moving parallel to a fixed plane, which we shall call the xy-plane (Art. 244). Let the coördinates of G, the center of gravity of the body referred to the *fixed axes* OX, OY be $\bar{x}$, $\bar{y}$, and the coördinates of any particle at P referred to the fixed axes be x, y, and the coördinates P referred to *axes through* G, *parallel to the fixed axes*, be x', y'.

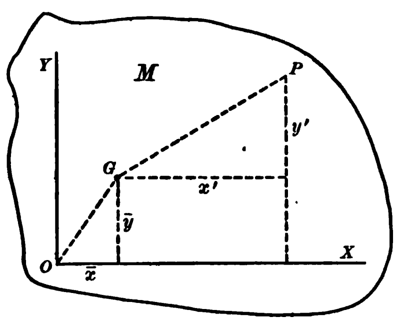


FIG. 143

Then

$$x = \bar{x} + x'; \quad \therefore \frac{dx}{dt} = \frac{d\bar{x}}{dt} + \frac{dx'}{dt};$$

$$\text{or } v_x = \bar{v}_x + v'_x.$$

$$y = \bar{y} + y'; \quad \therefore \frac{dy}{dt} = \frac{d\bar{y}}{dt} + \frac{dy'}{dt};$$

$$\text{or } v_y = \bar{v}_y + v'_y,$$

where v, $\bar{v}$ are respectively the velocity of the particle and the velocity of the center of gravity with respect to fixed axes,

and v' the velocity of the particle with respect to parallel axes through the center of gravity.

The kinetic energy E of the particle P whose weight is dW is

$$E = \frac{1}{2}\frac{dW}{g} \cdot v_i^2;$$

but

$$v_i^2 = v_x^2 + v_y^2 = (\bar{v}_x + v'_x)^2 + (\bar{v}_y + v'_y)^2,$$

$$= \bar{v}_x^2 + \bar{v}_y^2 + v'^2_x + v'^2_y + 2\,\bar{v}_x v'_x + 2\,\bar{v}_y v'_y.$$

Now

$$(\bar{v}_x^2 + \bar{v}_y^2) = \bar{v}^2;$$

$$v'^2_x + v'^2_y = v'^2.$$

Also

$$\sum \frac{dW}{g}\bar{v}_x v'_x = \bar{v}_x \sum \frac{dW}{g} v'_x = 0;$$

for the origin of the coördinates x', y' is the center of gravity of the body. Hence,

$$\sum \frac{dW}{g} x' = \frac{W}{g}\bar{x} = 0.$$

$$\therefore \sum \frac{dW}{g}\frac{dx'}{dt} = 0 = \sum \frac{dW}{g} v'_x.$$

Similarly,

$$\bar{v}_y \sum \frac{dW}{g} v'_y = 0.$$

Hence summing for all particles of the body

$$E = \sum \frac{dW}{2g} v_i^2 = \sum \frac{dW}{2g} \cdot \bar{v}^2 + \sum \frac{dW}{2g} v'^2.$$

Since however P is at a fixed distance from G, we may write as in (2) of this article, if $\omega =$ the angular velocity of the body and I the moment of inertia of the body about an axis through G perpendicular to the fixed plane,

$$\sum \frac{W_i}{2g} v'^2_i = \frac{1}{2} I\omega^2,$$

$$\therefore \frac{1}{2}\frac{Wv^2}{g} = \frac{1}{2}\frac{W\bar{v}^2}{g} + \frac{1}{2} I\omega^2. \quad . \quad . \quad . \quad . \quad (102)$$

Therefore, the kinetic energy of any body moving parallel to a fixed plane = the kinetic energy of a particle, at the center of gravity of the body and with a mass equal the mass of the body + the kinetic energy of the body rotating about an axis through the center of gravity of the body perpendicular to the fixed plane.

There is an analogous theorem for the kinetic energy of any rigid body, viz.: The kinetic energy of a body is

$$E = \frac{1}{2}\frac{W}{g}\bar{v}^2 + \frac{1}{2}I\omega^2,$$

where $\frac{W}{g}$ is the mass of the body, $\bar{v}$ the velocity of the center of gravity of the body, ω the angular velocity of the body about an axis through the center of gravity parallel to the instantaneous axis of rotation, and I the moment of inertia of the body about the same axis. The demonstration of this theorem carries us beyond the limits we have set for ourselves.

266. Work and Kinetic Energy. — Since $a_{x_i} = \frac{v_{x_i}dv_{x_i}}{dx_i}$ the ith equation of the first column of the equation (92) can be written

$$\frac{W_i}{g} \cdot v_{x_i}dv_{x_i} = Xdx_i + X'dx_i.$$

Treating each of the three of the ith equations of (92) in the same way, summing and integrating, we get, since the sum of the work of the internal forces is zero,

$$\sum\frac{W_i}{2g}(v_{x_i}^2 + v_{y_i}^2 + v_{z_i}^2) = \sum\frac{W_i}{2g}v_i^2 = \int(Xdx + Ydy + Zdz) + C.$$

If we assume that $v_i = v_0$ when $x = x_0$, $y = y_0$, $z = z_0$, we obtain

$$\sum\frac{W_i}{2g}(v_i^2 - v_0^2) = \int_0^{xyz}(Xdx + Ydy + Zdz), \quad (103)$$

i.e. *the change in the kinetic energy of a rigid body is equal to the work done by the external forces.*

If the body is in translation, since velocity of all the particles is the same, v_i may be taken from under the summation sign, and equation (103) becomes

$$\frac{W}{2g}(v^2 - v_0^2) = \int (Xdx + Ydy + Zdz),$$

where W is the weight of the body. Suppose that $Y = Z = 0$, then the acting force is parallel to x-axis, and equation (103) becomes

$$\frac{W}{2g}(v^2 - v_0^2) = \int Xdx = \text{the work of the external forces.}$$

If the body is rotating about a fixed axis, the z-axis say, $v_i = r_i\omega$, where r_i is the distance of the particle from the fixed axis and ω the angular velocity of the body about the axis, equation (103) now becomes

$$I_z\frac{(\omega^2 - \omega_0^2)}{2} = \int (Xdx + Ydy) = \text{the work of the external forces.}$$

If the body is rotating about an axis which is itself translating, then (Art. 265)

$$\frac{1}{2}\sum \frac{W}{g}(v^2 - v_0^2) = \frac{1}{2}\frac{W}{g}(\bar{v}^2 - \bar{v}_0^2) + \frac{1}{2}I(\omega^2 - \omega_0^2) = \text{work of the external forces.}$$

To illustrate: Let us consider the problem.

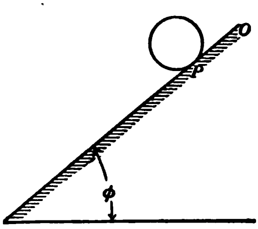


FIG. 144

267. *Problem.* — To determine the motion of a homogeneous right cylinder rolling down a perfectly rough plane, inclined an angle ϕ to the horizon. Let the weight of the cylinder be W, and its radius r. Suppose it start from rest from O, and at any time t let the point of contact of the cylinder with the plane be P, and at that time let the distance that the center of the cylinder has traveled equal s. Then the angle that a radius of the cylinder

has rolled through equals $\frac{s}{r} = \theta$. Let the velocity of the center of the cylinder be v; the angular velocity of a radius will then be $\frac{v}{r} = \omega$.

Similarly a and $\alpha = \frac{a}{r}$ equal the linear and angular acceleration respectively.

The work done by the weight of the cylinder in moving a distance s down the plane equals $W \cdot s \sin \phi$.

The change in kinetic energy, since the initial velocity is zero, is, from Art. 265,

$$= \frac{1}{2} \frac{W}{g} \cdot v^2 + \frac{1}{2} \omega^2 I.$$

Then, from Art. 266,

$$Ws \sin \phi = \frac{1}{2} \frac{W}{g} \cdot v^2 + \frac{1}{2} \omega^2 I.$$

But

$$I = \frac{W}{g} \cdot \frac{r^2}{2} \text{ and } \omega = \frac{v}{r};$$

$$\therefore W \cdot s \sin \phi = \frac{1}{2} \frac{W}{g} \cdot v^2 + \frac{1}{2} \cdot \frac{v^2}{r^2} \cdot \frac{W}{g} \cdot \frac{r^2}{2};$$

$$Ws \sin \phi = \frac{W}{g} \left(\frac{1}{2} v^2 + \frac{1}{4} v^2 \right). \quad . \quad . \quad . \quad . \quad (a)$$

$$\therefore v = \sqrt{\frac{4g \cdot s \sin \phi}{3}} \text{ and } \omega = \sqrt{\frac{4g \cdot s \sin \phi}{3r^2}}.$$

From equation (a) it is evident that $\frac{1}{3}$ of the work done is used to turn the cylinder and $\frac{2}{3}$ to urge it down the plane.

To find the acceleration, we have

$$vdv = ads. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Noting that $v = 0$, when $s = 0$ upon integration we obtain

$$\frac{v^2}{2} = as.$$

$$\therefore as = \frac{2g}{3} \cdot s \sin \phi,$$

and $$a = \frac{2g}{3} \sin \phi.$$

The angular acceleration must then be

$$\alpha = \frac{2g}{3r} \sin \phi.$$

268. Exercises

1. If instead of a cylinder as in Art. 267, a sphere lies on a rough plane inclined an angle ϕ to the horizon, find its motion.

2. If the sphere in exercise 1 has a radius of 18 inches, find at the end of 5 seconds:

(*a*) How far its center of gravity has moved.

(*b*) The velocity of the center of gravity.

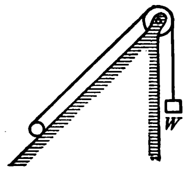
(*c*) The angular velocity about an axis through P (see Fig. 143).

(*d*) The angular velocity about an axis through the center of gravity.

3. A grindstone weighing 175 pounds is making 180 revolutions per minute. If the radius of the stone is 18 inches and there is a tangential friction of 5 pounds on the circumference of the stone, how many revolutions will the stone make before coming to rest? Neglect journal friction.

4. A wire cable weighing 4 pounds per foot is wound around a drum, whose radius is 3 feet. Assume that at the beginning 5 feet of the cable is unwound, and that the cable does not affect the motion until it is unwound. If the radius of gyration of the drum is 2.75 feet, and the weight of the drum 200 pounds, neglecting all friction, what would be the angular velocity of the drum when 105 feet of the cable are unwound? What would be the linear acceleration of the end of the cable?

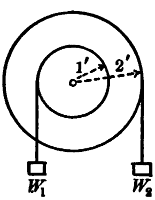
5. A solid cylinder, having a radius of 1 foot and weighing 500 pounds, is rolled up a plane, inclined 45° to the horizon, by a cable which is wrapped around it and attached to a weight



as shown. Neglect the weight of the cable and pulley, and assume that the plane is rough enough to produce perfect rolling. Find the weight of the suspended body in order that the center of the cylinder may have a velocity of 5 feet per second at the end of 15 seconds.

6. In exercise 5, let the cylinder be replaced by a sliding block of the same weight; the plane now smooth. Find the weight of the suspended body in order that the block may have the same velocity as the cylinder in exercise 5.

7. In exercise 6, how much would the velocity be decreased if the weight of the pulley were considered? Assume its weight to be 40 pounds, and its radius of gyration to be 9 inches.



8. Two solid cylinders whose weights are 400 pounds and 800 pounds, and whose radii are 1 foot and 2 feet respectively, are mounted on axles coinciding with their geometrical axes. There are wound around these cylinders cables, considered weightless, to which are attached weights, $W_1 = 250$ lb. and $W_2 = 300$ lb. If the system is initially at rest, and the friction neglected, find:

(*a*) The velocity of each weight when W_2 has moved 12 feet.

(*b*) The angular velocities of the cylinders at that time.

(*c*) The angular acceleration of the cylinders and the linear acceleration of the weights at that time.

(*d*) The time of the descent.

9. A locomotive can exert a pull of 20,000 pounds on the drawbar. Suppose that the weight of the train is 540 tons, that there is a constant friction of 12 pounds per ton, that the grade is 1 foot vertical to 100 feet horizontal, and that the velocity is 10 miles per hour. Find the velocity at the end of 1000 feet up the slope, first neglecting the work necessary to rotate the wheels; second, considering that there are 9 cars and 6 pairs of wheels to the car, the weight of each pair being 1500 pounds, radius 15 inches and radius of gyration 7.5 inches.

CHAPTER XVI

KINETIC FRICTION

269. In the present chapter we shall deal with the friction of blocks sliding on a fixed surface; the friction of axles in their bearings; the friction of pivots; the friction of belts; and with what is commonly called rolling friction. Static sliding friction was considered in Art. 109 and those that follow.

270. Laws of Kinetic Friction. — Our knowledge regarding "kinetic" friction is somewhat imperfect. There are so many variables entering that any general statement will not always hold. In any case of great importance, experiments should be made under the actual working conditions to determine the amount of the friction and the manner in which it varies with the speed, pressure, and lubricants. The following laws stated by Archbutt and Deeley in *Lubrication and Lubricants*, page 12, are perhaps as near the truth as any statements of their length can be:

1. The frictional resistance is approximately proportional to the load on the rubbing surface.

2. The frictional resistance is slightly greater for large areas and small pressures than for small areas and great pressures.

3. The frictional resistance, except at very low speeds, decreases as the velocity increases.

271. Coefficients of Friction. — The coefficient of kinetic friction is defined in the same way as the coefficient of impending friction, *i.e.*

$$f = \frac{F}{N} = \tan\phi,$$

where $f =$ coefficient of kinetic friction,

$F =$ total friction,

$N =$ the normal pressure.

The following coefficients are average values for poorly lubricated surfaces. They may be used to solve problems in the text.

Wood on Wood	0.35
Steel on Steel — Dry	0.14
Cast Iron on Cast Iron	0.15
Stone on Stone	0.6 to 0.7

272. An Experiment in Sliding Friction. — An experiment for determining the coefficient of limiting friction was described in Art. 115. A block was placed on the lid of a desk and the angle between the lid and the horizontal was increased until the block was on the point of moving. Let us assume that the angle is θ when the block slides down the plane. The forces acting on the block are W, its weight, F, the frictional component, N, the normal component of the pressure between the block and the plane, and P, the resultant of all other external forces. Then, if the velocity of the block down the plane is uniform, we have, summing the components parallel to the plane,

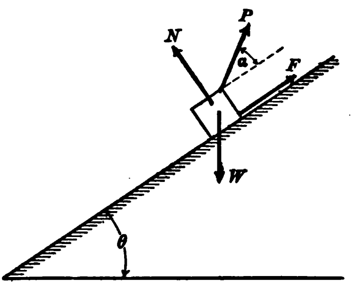


Fig. 145

$$P \cdot \cos \alpha - W \cdot \sin \theta + F = 0. \quad (a)$$

Summing the components perpendicular to the plane, we have

$$N - W \cdot \cos \theta + P \cdot \sin \alpha = 0. \quad (b)$$

Since $f = \frac{F}{N}$, we obtain, by solving equations (a) and (b),

$$f = \frac{W \sin \theta - P \cos \alpha}{W \cos \theta - P \sin \alpha}. \quad (c)$$

An expression for f may be obtained if the velocity of the block down the plane is variable. It will be assumed that the friction is so nearly constant that the motion can be considered uniformly accelerated. Let the distance covered by the block, from a state of rest, in t seconds, be s feet, s being measured from the point when t is zero. The forces acting on the block are the same as before. The sum of the components parallel to the plane is however equal to $\frac{W}{g} \cdot a$ instead of zero, since the motion is uniformly accelerated.

$$\therefore\ W \sin\theta - P\cos\alpha - F = \frac{W}{g} \cdot a. \quad . \quad . \quad . \quad . \quad (d)$$

Since there is no motion perpendicular to the plane, the sum of the components perpendicular to the plane will equal zero as before. Hence

$$N - W\cos\theta + P\sin\alpha = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

The acceleration being constant,

$$s = \tfrac{1}{2} at^2;$$

and

$$a = \frac{2s}{t^2}.$$

Substituting for a and solving for F in equation (d), we obtain

$$F = W\sin\theta - P\cos\alpha - \frac{2W}{g} \cdot \frac{s}{t^2}.$$

$$N = W\cos\theta - P\sin\alpha.$$

$$\therefore f = \frac{F}{N} = \frac{W\sin\theta - P\cos\alpha - \frac{2W}{g}\frac{s}{t^2}}{W\cos\theta - P\sin\alpha}.$$

If there is no force P acting, f becomes

$$f = \tan\theta - \frac{2s \cdot \sec\theta}{gt^2}.$$

273. A Second Experiment in Sliding Friction. — A weight W_1 is moved over a rough horizontal table by a string AB, which is parallel to the plane, and which passes over a smooth surface at B and is attached to a weight W_2. Find the tension T of the string and the friction F between the block and the table, assuming that the acceleration a, of W_1, is uniform, and that the string is flexible and weightless.

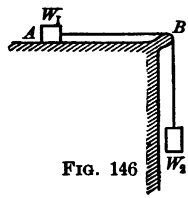


FIG. 146

Consider first W_2; the forces acting on it are T and W_2. Hence,

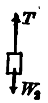


FIG. 146 *a*

$$W_2 - T = \frac{W_2}{g} \cdot a.$$

Consider now W_1. The forces are T, W_1, N, the normal component, and F, the frictional component of the reaction of the table on the block. Taking the components parallel to the table,

$$T - F = \frac{W_1}{g} \cdot a.$$

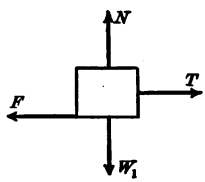


FIG. 146 *b*

We leave as an exercise for the student:

1. Having observed that W_1 has traveled from rest s feet in t seconds, to find f.

2. To find $T = \frac{W_1 W_2}{W_1 + W_2}(1 + f)$.

3. To find the velocity of W_1 at the end of t seconds.

274. Exercises

1. The block of Fig. 144 weighs 32.2 pounds; f, the coefficient of kinetic friction, is 0.35 and P is 0; find the value of θ that the motion down the plane may be uniform.

2. For what value of θ in exercise 1 will the block starting from rest move 73 feet in 4 seconds?

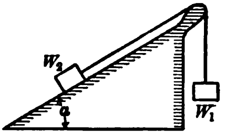
3. Let W_1 and W_2 in Fig. 146 equal 64.4 pounds each, with the apparatus arranged as in that figure. What is the value of the coefficient of friction if the block W_2 descends 60 feet in 3 seconds, from rest?

4. A wooden box weighing 50 pounds contains scrap iron. The coefficient of friction between the box and the floor is 0.25. If a pull of 75 pounds must be exerted upon a rope fastened to the box and running parallel to the floor, in order that the box may move with a uniform velocity of 2 feet per second, how many pounds of iron does the box contain?

5. A grindstone weighing 300 pounds is being dragged over the cement floor of a shop by a rope which makes an angle of 30° with the floor. What is the pull on the rope, if $f = 0.6$ and the velocity is uniform?

6. What angle with the floor should the rope in exercise 5 make that the pull on the rope may have the smallest possible value?

7. A piece of cast iron weighing 100 pounds is rigidly fastened to the table of a small planer. The table and bolts for fastening the work to the table weigh 125 pounds. If the coefficient of friction is 0.07, how much power is expended in overcoming friction when the planer is making three complete strokes per second, the length of a single stroke being 4 feet?



8. The weights W_1 and W_2 are connected by a flexible inextensible cord as shown. W_2 weighs 100 pounds and the coefficient of friction between it and the plane is 0.20. If α is 30°, what must be the weight of W_1 that W_2 may move up the plane with an acceleration of 2 feet per second per second?

9. Find the amount of the weight of W_1 of exercise 8, if W_2 moves down the plane with an acceleration of 2 feet per second per second.

275. Axle Friction. A Single Element of Contact.—If a horizontal cylindrical shaft, S, rests loosely in a horizontal cylindrical bearing, B, only a very narrow strip of each of the cylinders is in contact; in fact, it is usually assumed that the surface of contact is a single element of the cylindrical surface.* Let it be required to find the position of this element of contact, when the shaft rotates in the bearing, and friction exists between the shaft and the bearing. Let us assume that all forces which tend to cause the shaft to rotate, such as belt pulls or pressures of gear teeth, have a resultant P, and that all resistances to rotation, the weight of the shaft and all other forces except R, the reaction of the bearing on the shaft and P, have a resultant Q. Then as the shaft begins to rotate, it will roll in the loose bearing until a position is reached such that the friction is exceeded and slipping begins.

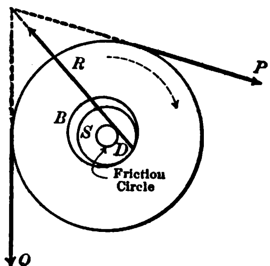


FIG. 147

To find this position, let us apply graphical methods since they are more simple and direct than the analytic method. First, the reaction of the bearing upon the shaft must make an angle ϕ, where ϕ is the angle of friction, with the radius of the shaft at the point of contact regardless of where that point may be. That is, the length of the perpendicular dropped from the center of the shaft to the line of the reaction of the bearing must be $r \sin \phi$, where r is the radius of the shaft. To accomplish this graphically, draw a circle with $r \sin \phi$ as radius; then the line of action of R must be tangent to this circle. *This circle is called the friction circle*, and its use greatly simplifies all work in axle friction. Secondly, the line of action of R must pass through the intersection of P and Q. Therefore, if we draw a line through the intersection of P and Q and tangent to the friction circle on the side which opposes the motion, the point of intersection of this line with the bearing is the point of contact.

* See, however, Art. 277.

The amount of friction is

$$F = R \sin \phi,$$

or, as it is more commonly written,

$$F = R \tan \phi = fR.$$

The second form is used because of the slight difference between the sines and tangents of small angles and because of the uncertainty of the values of f.

The work per second, that is, the power necessary to overcome the friction when the shaft is making n revolutions per second is

$$\text{Power} = F \cdot 2\pi r \cdot n = R \sin \phi \cdot 2\pi r \cdot n$$

or $$\text{Power} = fR \cdot 2\pi r \cdot n.$$

276. Axle Friction. Shaft supported at Two Points. — Let us assume that the shaft of the preceding article is supported at two points, as in Fig. 148; to find the amount of the friction. In this case each of the reactions makes the angle ϕ with the radius at the point of contact. Therefore, we will draw R' and R'' from the points of contact tangent to the friction circle on the side opposed to the motion, until they intersect as at M. Let P and Q represent the forces causing rotation and resistance to rotation respectively, as in the preceding article, and let P and Q intersect at N. Then the line of action of the resultant of R' and R'' is given by the line MN. The amount of the resultant is found by completing the parallelogram on P and Q, thus obtaining the resultant R. Now the resultant of R' and R'' must be equal and opposite and lie in the same line as R. Therefore we can lay off a distance from M equal to R, and the sides of the parallelogram obtained by drawing

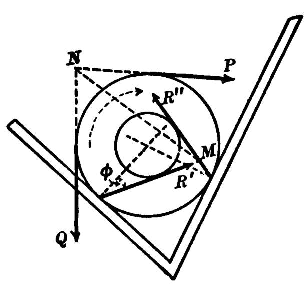


Fig. 148

lines through the extremities of R parallel to R' and R'' respectively, will give their values.

The amount of the friction, F, is

$$F = R' \sin \phi + R'' \sin \phi.$$

Replacing $\sin \phi$ by $\tan \phi$ as before, we have

$$F = f(R' + R'').$$

The power required to overcome the friction when the shaft makes n revolutions per second is

$$\text{Power} = f(R' + R'')\, 2\,\pi r \cdot n.$$

277. Axle Friction. Arc of Contact.—When the shafts and their bearings are properly fitted, there appears to be an arc of contact between each shaft and its bearing such as the arc AB in the figure. It has been shown, however, that in a properly lubricated bearing, the shaft is actually not in contact with the bearings, but is separated from it by a thin film of oil. The friction which resists the turning of the shaft is due to the viscosity of the oil, rather than to the rubbing of the axle on the bearing. Let us determine the power required to overcome this friction.

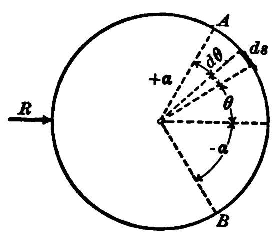


FIG. 149

The intensity of pressure over the arc AB is assumed to be constant, although it is undoubtedly true that there is a variation of pressure due to the tendency of the shaft to climb in the bearing as suggested in Art. 275. The pressure on each element of the surface $ds = pds$, and its horizontal component $= pds \cdot \cos \theta$, but $ds = rd\theta$.

$$\therefore\ R = pr \int_{-\alpha}^{+\alpha} \cos \theta d\theta = 2\, pr \sin \alpha$$

and $p = \dfrac{R}{2\, r \sin \alpha}$, where α equals $\dfrac{1}{2}$ the angle of contact.

Therefore the friction on each element equals

$$dF = pds \cdot \tan \phi = fpds$$

since the total reaction on each element will make an angle ϕ with the radius of the shaft at the point of contact, exactly as R' and R'' did in the last article.

$$\therefore\ F = \int fpds = fp\int_{-\alpha}^{+\alpha} rd\theta = 2\,fpr\alpha.$$

Substituting the value of p, we have

$$F = \frac{2\,fRr\alpha}{2\,r\sin\alpha} = \frac{fR\alpha}{\sin\alpha}.$$

When the arc of contact is small $\alpha = \sin\alpha$ and $F = fR$, as in Art. 275, where only one element is in contact.

When the arc of contact is 180°, as it is usually assumed in practice,

$$F = \frac{fR\pi}{2} = f'R.$$

The power lost when the shaft is making n revolutions per second and the arc of contact is 180° is

$$\text{Power} = f'R \cdot 2\,\pi rn.$$

The student will note that the intensity of pressure p equals the total pressure R, divided by the projection of the arc of contact upon a plane normal to the line of action of R; or when the arc of contact is 180°, it equals the total pressure divided by the diametral area. The intensity of pressure allowed in practice is determined by experiment so that the bearing may move smoothly and avoid any danger of heating.

The values of f' are determined by experiment. There is such a wide variation with different methods of lubrication, different pressures and different velocities, that the reader is referred to more extended treatises such as Archbutt and Deeley's *Lubrication and Lubricants*, for a more complete discussion and for values of the coefficient under different conditions. The student may use $f' = 0.05$ for the exercises which follow.

278. **Exercises**

1. The belt pull on each side of an idle pulley 16 inches in diameter is 800 pounds, and the angle of contact is 60°. If the radius of the axle is one inch, what is the power lost when the pulley is making 240 R. P. M.?

2. What is the amount of the power necessary to overcome the axle friction in the 12 wheels of a passenger coach which with its load weighs 100,000 pounds, the velocity of the train being 60 M. P. H.? The diameter of the wheels is 30 inches; the diameter of the axles, 2 inches.

3. Assuming that the coefficient of static axle friction is $f = 0.10$, what is the greatest pull that could be applied tangent to the circumference of a wagon wheel four feet in diameter before the wheel begins to revolve? Assume that the weight of the wheel with its load is 1000 pounds, that the radius of the axle is 1 inch, and that the only resistance to be overcome is the friction.

279. Pivot Friction. — When a shaft is vertical or inclined, and sometimes when horizontal, there is a thrust along the shaft which must be taken by some kind of a bearing. The entire end of the shaft may press against a bearing like a pivot, or the bearing may be arranged so that a ring only comes in contact, or the thrust may be carried by a collar upon the shaft. In any case, the friction between the part of the shaft in contact with the bearing surface is called *pivot friction* and the work necessary to overcome it may be found as follows:

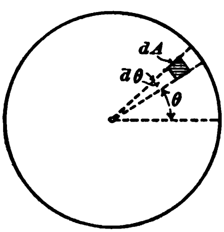


FIG. 150

Let us assume that the flat end of a cylindrical shaft is in contact with the bearing over its entire end, and that the intensity of the bearing pressure p is constant. Then the total pressure on the differential area dA is pdA and the friction upon that element is $fpdA$. The

work dw necessary to overcome this friction during one revolution is

$$dw = fpdA\, 2\,\pi \cdot \rho,$$

where ρ is the distance of the element from the center of the pivot. Using polar coördinates,

$$dA = \rho d\theta \cdot d\rho.$$

Substituting, we have

$$w = \int dw = 2\,\pi fp \int_0^{2\pi} \int_0^r \rho^2 d\rho \cdot d\theta.$$

Integrating,

$$w = \frac{2\,\pi f p r^3}{3} \int_0^{2\pi} d\theta = \frac{4\,\pi^2 f p r^3}{3}.$$

The total thrust, T, of the shaft is

$$T = \pi r^2 \cdot p.$$

Therefore the work of one revolution $= \frac{4}{3}\,\pi f T r$; the power required to overcome the friction for n revolutions per second, is

$$\text{Power} = \tfrac{4}{3}\,\pi f r n T. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

If the shaft were hollow so that the section in bearing was a ring section, the outer radius being r_1 and the inner r_2, the power necessary to overcome the friction when the shaft is making n revolutions per second is

$$\text{Power} = \frac{4}{3}\pi f n \frac{(r_1^3 - r_2^3)}{(r_1^2 - r_2^2)} T. \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

If the end of the shaft is conical instead of flat, the intensity of pressure becomes $\dfrac{p}{\sin \alpha}$, where α is the angle between the axis of the shaft and the conical surface. If we let $r' =$ radius of the cone at top of bearing and proceed as before, the total power required to overcome the friction is

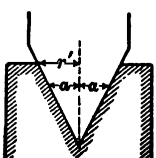


FIG. 151

$$\text{Power} = \frac{4}{3} \frac{f \pi r' n T}{\sin \alpha}. \quad . \quad . \quad . \quad . \quad (c)$$

It is evident from equation (c) that for conical pivots with small values of α, the work lost by friction is enormous.

280. Schiele's Antifriction Pivot. — As is evident from Fig. 151, the conical pivot will wear most rapidly along a ring farthest from the axis of rotation, and consequently in a time depending on the normal pressure, the hardness of the material, and the speed, there will not be perfect contact between the pivot and its step. If, however, the curve made by a plane section through the common axis of pivot and step is such that the wear on the pivot parallel to the axis of rotation is the same for all points on the curve, the pivot and step will remain in perfect contact. Let us find the equation of such a curve.

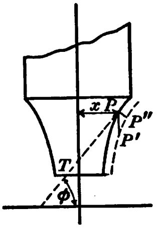


FIG. 152

Let P be a point on the pivot in the xy-plane. Suppose the step to have worn in the form of the dotted curve. Let PP'' be the normal wear, and PP', the vertical wear. Let f be the coefficient of friction and p the normal pressure per unit of area, and let θ be the angle through which the pivot has turned. Then the normal wear on a unit area at a distance x from the axis of rotation will be proportional to the work done by the friction on that unit area. That is,

$$PP'' = kfp\theta x,$$

where k is a constant depending on the hardness of the materials.

Now the wear PP' is assumed to be the same for all points on the curve. But if ϕ is the angle that the tangent to the curve at P makes with the x-axis, P being in the xy-plane,

$$PP' = \frac{PP''}{\cos\phi} = \frac{k \cdot fp\theta x}{\cos\phi} = \text{a constant};$$

$$\therefore \frac{x}{\cos\phi} = \frac{x}{\dfrac{dx}{ds}} = \text{a constant.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

That is, if T be the point where a tangent to the curve at P cuts the y-axis,

$$PT = \text{a constant}.$$

This curve is known as the tractrix.

Its equation may be found by substituting the value $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ in (a) and integrating. If we put $x \cdot \frac{ds}{dx} = a$, the equation of this curve will be found to be

$$a \log\left(\frac{a - \sqrt{a^2 - x^2}}{x}\right) + y + \sqrt{a^2 - x^2} = 0.$$

This form of the pivot is known as *Schiele's Antifriction Pivot.*

281. Exercises

1. A vertical shaft carrying a load of 5 tons is 4 inches in diameter and makes 120 revolutions per minute. Find the work lost per revolution if the pivot is flat and the coefficient is 0.1. What is the horse-power lost?

2. What horse-power would be lost if the pivot in exercise 1 were conical, α being 30° and r' $1\frac{7}{8}$ inches? (See Fig. 151.)

3. A horizontal shaft is subject to a thrust of 150 tons. The shaft is 18 inches in diameter and the thrust is taken by a collar whose outside diameter is 26 inches. Find the work lost per revolution if $f = 0.08$, and the power lost when the shaft is making 60 revolutions per minute.

4. What horse-power would be lost if a Schiele's antifriction pivot were used in exercise 1?

282. Rolling Friction. — When a cylinder rolls upon an imperfectly elastic surface, there is a resistance to be overcome due to the fact that the surface is compressed in front of the cylinder, and in effect the cylinder is constantly rolling up hill. On account of this compression the point of application of the reaction of the surface upon the cylinder falls a small

distance ahead of the vertical through the center of the cylinder. *This distance is called the coefficient of rolling friction,* although it is not a coefficient in the sense that f is the coefficient of sliding friction, but is actually a distance, ordinarily measured in inches.

The theory of rolling friction is at present so uncertain that we shall not go into it further, but will only give the following values of the coefficient of rolling friction as reported in the *American Civil Engineer's Hand Book,* p. 1223.

Lignum Vitæ on Oak Track	0.019 inch
Elm Roller on Oak Track	0.032 inch
Cast-iron Wheel (20 in. diam.) on Cast-iron Rail	0.018–0.019 inch
Railroad Wheels (39.4 in.)	0.020–0.022 inch
Iron or Steel Wheels on Wood Track	0.06–0.10 inch

283. Antifriction Wheels. — Suppose an axle A of radius r, carrying a load W, rests upon two wheels, B and B', each of radius r_1, instead of in a bearing. Problem: Is the friction increased or diminished? Let β equal the angle between a vertical through the center of the axle A and a line joining the center of axle A and the center of the axles of the so-called antifriction wheels B and B'. Then the weight carried by B and B' equals $\frac{W}{2\cos\beta}$; the friction of each axle, B and B', equals $\frac{fW}{2\cos\beta}$.

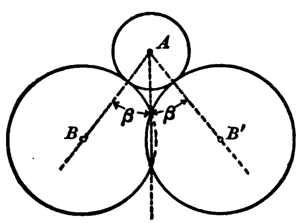


FIG. 153

The power necessary to overcome the friction at both axles when the axle A is making n revolutions per second is

$$\text{Power} = 2 \cdot \frac{fW}{2\cos\beta} \cdot 2\,\pi r_2 \cdot \frac{nr}{r_1},$$

where r_2 is the radius of the axle of the friction wheels.

The power necessary, if the axle A were in an ordinary bearing, slightly worn, is

$$\text{Power} = fW \cdot 2\,\pi \cdot n \cdot r.$$

Therefore there will be a saving of energy if

$$\frac{r_2}{r_1 \cos \beta} > 1.$$

This result shows that for values of β of 45° or less, a large saving can easily be effected.

For a treatise on roller bearings, ball bearings, etc., the student is referred to Kent's *Pocket Book for Mechanical Engineers*, p. 1210.

284. Belt Friction. — The ability of a belt to transmit power is due to the friction between the belt and the pulley. Before any power can be transmitted, there must be tension in the belt on both sides of the pulley and the difference between the tension T_1 in the tight or driving side, and the tension T_2 in the loose or following side, is equal to the friction between the belt and the pulley.

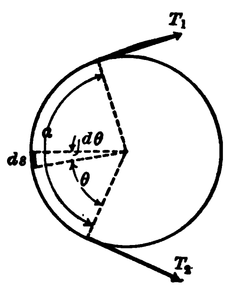


FIG. 154

We will now deduce a relation between T_1 and T_2 (Fig. 154). Consider "free" an element of the belt, ds. The forces acting on this element are the tension T in the belt on the side toward the following side; $T + dT$, the tension on the side toward the driving side; the friction on that element, dF; the pressure between the belt and the pulley, pds, where p is the intensity of pressure at that point; and the centrifugal force $\frac{cds}{g}\frac{v^2}{r}$, where c equals the weight of the belt per linear foot, v the velocity of the belt in feet per second, and r the radius of the pulley. Assuming that pds bisects the angle $d\theta$, and summing the components parallel to pds, after replacing ds by $rd\theta$, we have

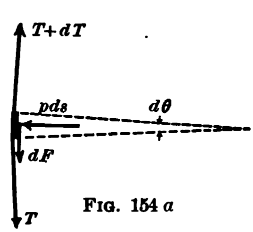


FIG. 154 a

$$(T+dT)\sin\frac{d\theta}{2}+T\sin\frac{d\theta}{2}=p\cdot rd\theta+\frac{c}{g}\cdot\frac{v^2}{r}\cdot ds.$$

$$\therefore p=\frac{(2\,T+dT)\sin\dfrac{d\theta}{2}}{rd\theta}-\frac{cv^2}{gr}.$$

But
$$\frac{\sin\dfrac{d\theta}{2}}{d\theta}=\frac{1}{2}.$$

Therefore,
$$p=\frac{T-\dfrac{cv^2}{g}}{r}.$$

Taking moments about the center of the pulley (Fig. 154 *a*),

$$(T+dT)r-Tr-dFr=0.$$

$$\therefore dT=dF,$$

but $dF=fpds=fprd\theta$ when $f=$ the coefficient of friction between the belt and the pulley. Replacing p by $\dfrac{T-\dfrac{cv^2}{g}}{r}$, we have

$$dT=f\left(T-\frac{cv^2}{g}\right)d\theta,$$

or

$$\int_{T_2}^{T_1}\frac{dT}{T-\dfrac{cv^2}{g}}=f\int_0^{\alpha}d\theta.$$

Integrating, we obtain,

$$\log_e\left(T_1-\frac{cv^2}{g}\right)-\log_e\left(T_2-\frac{cv^2}{g}\right)=f\alpha.$$

Therefore,
$$\frac{T_1-\dfrac{cv^2}{g}}{T_2-\dfrac{cv^2}{g}}=e^{f\alpha}.$$

For small belt speeds, the effect of the centrifugal forces is negligible and the quantity $\dfrac{cv^2}{g}$ can be neglected. Then,

$$\frac{T_1}{T_2}=e^{f\alpha}.$$

285. Power Transmitted by a Belt. — Given a pulley with a radius of r feet making n revolutions per minute; to find the power transmitted when the tension in the driving side of the belt is T_1 pounds and in the following side is T_2 pounds. Let F equal the total friction between the pulley and the belt. Then the power being transmitted equals

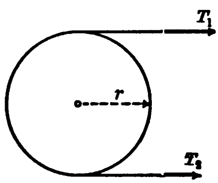


FIG. 155

$$F \cdot 2\,\pi r n.$$

Taking moments about the center of the pulley (Fig. 155 *a*),

$$T_1 r - T_2 r - F r = 0.$$

$$\therefore F = T_1 - T_2.$$

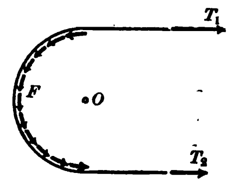


FIG. 155 *a*

Again v, the velocity of the belt, is

$$v = 2\,\pi r n.$$

$\therefore$ Power $= (T_1 - T_2)v$ foot-pounds per minute

$$= \frac{(T_1 - T_2)v}{33{,}000} \text{ horse-power.}$$

286. Transmission Dynamometers. — The power which is being transmitted by a belt can be measured by devices called Transmission Dynamometers. Figure 156 is a diagrammatic sketch of the Tatham dynamometer,* which measured the power consumed by the dynamo-electrical machines tested by a committee of judges in June, 1885 (see report in supplement to the *Franklin Institute Journal*, Nov., 1885). The dynamometer was capable of measuring 100 horse-power; the largest machine then required 70 horse-power; the smallest, 0.23 horse-power.

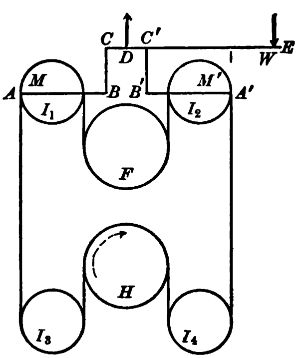


FIG. 156

* From *Dynamometers and the Measurement of Power*, Flather, p. 95.

The dynamometer consists of a driving pulley at H, and the pulley F which is connected to the machine whose power is to be measured. I_1, I_2, I_3, and I_4 are idle pulleys; I_3 and I_4 being arranged with micrometers so that by moving them, any desired tension in the belt may be secured. I_1 and I_2 are attached at M and M' to exactly similar levers whose fulcrums are at A and A'. The levers are attached to a second lever at C and C', the fulcrum of which is at D and upon which is a weight W whose distance from D can be varied at will. Let us now consider the lever ABC. We have acting at M on this lever the force from the pulley, equal to $2\,T_1$ where T_1 is the tension in the driving side of the belt, and at C, S_1 the pull of the lever ABC on the lever DE, and at A the reaction R' of the fulcrum.

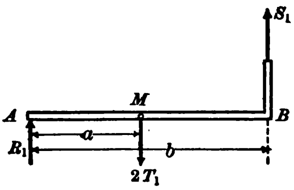


FIG. 156 a

Taking moments about A,

$$S_1 \cdot b = 2\,T_1 \cdot a.$$

$$\therefore\ S_1 = \frac{2\,T_1 \cdot a}{b}.$$

Similarly, with lever $A'B'C'$,

$$S_2 = \frac{2\,T_2 \cdot a}{b},$$

where T_2 is the tension in the following side of the belt.

Let us now consider DE free and take moments about D.

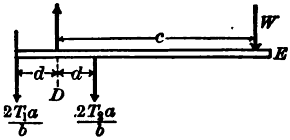


FIG. 156 b

Then

$$Wc = \frac{2\,T_1 \cdot ad}{b} - \frac{2\,T_2 \cdot a \cdot d}{b},$$

or

$$T_1 - T_2 = \frac{Wcb}{2\,ad}.$$

Evidently to measure the power delivered to a machine knowing W, a, b, and d, it is necessary to measure c, and the number of revolutions per second only.

287. **Exercises**

1. The tension on the following side of a belt is 180 pounds, the angle of contact 220°, and the coefficient of friction, 0.3. What is the horse-power transmitted if the velocity of the belt is 4500 feet per minute?

2. The allowable tension on the tight side of a single belt is 60 pounds per inch of width. How wide should a belt be to transmit 22 horse-power, with a belt speed of 3000 feet per minute, the angle of contact being 220°, and $f = 0.27$?

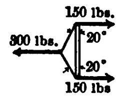
3. The tension on the tight side of a belt is 240 pounds, on the slack side 120 pounds; the angle of contact is 180°. What is the coefficient of friction, f?

4. What error would there be in neglecting the centrifugal force if the belt speed in exercise 3 is 10,000 feet per minute? Assume the weight of the belt per linear foot to be 1 pound.

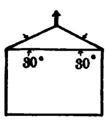
5. How many times must a deck hand wrap a cable around a cylindrical pierhead to slacken the motion of a ship if he pulls 10 pounds on the cable and the tension put on the cable by the ship is 3 tons? Assume that the coefficient of friction between the pierhead and the cable is 0.25.

MISCELLANEOUS PROBLEMS

1. A horse pulls 300 pounds on a whiffletree. Chains on the whiffletree make angles of 20° with the beam. Find the tension of the chains and the compression of the beam.



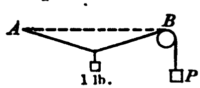
2. A man weighing 180 pounds lies in a hammock. The angle made by the hammock and the horizon at one end is 15° and at the other end is 30°. Find the tension on each end of the hammock.



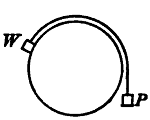
3. A picture, weighing 20 pounds, hangs on the wall by a wire which makes an angle of 30° with the horizontal edge of the frame. Find the tension of the wire.

4. Two telegraph wires, each in a horizontal plane, are under a tension of 250 pounds and are fastened to the top of a pole. The angle between the wires is 90°. On the opposite side of the pole in a plane that passes through the pole and the bisector of the angle between the wires is a stay wire which makes an angle of 45° with the ground. Find the tension on the stay wire and the downward pressure on the pole.

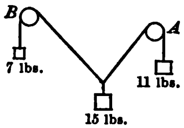
5. A cord is firmly fastened to a point A, then passes through a smooth movable pulley. Finally it passes over a smooth stationary pulley, B, AB horizontal, and is attached to a weight P of three pounds. A weight of one pound is attached to the movable pulley. Find the amount and direction of the pull on the fixed pulley.



6. A weight of W pounds is placed on a smooth cylinder and a cord passes over the cylinder to which is suspended a weight of P pounds. Find the position of equilibrium.

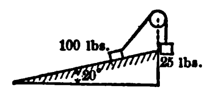


7. A and B are smooth fixed pulleys over which a cord is passed. On the end which passes over A is fastened a weight



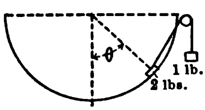
of 11 pounds, and on the end which passes over B is a weight of 7 pounds. Between A and B there is a movable pulley to which is fastened a weight of fifteen pounds. Find the angles which the two segments of the rope make with the horizon.

8. A plane making an angle of 20° with the horizon has a weight of 100 pounds resting upon it. A cord, fastened to this weight, passes over a pulley whose center is one foot above the plane and whose radius is four inches. A weight of twenty-five



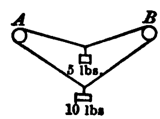
pounds is suspended on the end of the cord. How far down the plane from a vertical through the center of the pulley will the 100-pound weight rest when equilibrium is established?

9. Inside a smooth fixed hemispherical bowl is placed a two-



pound weight. A cord is fastened to the weight and passed over the rim of the bowl, which is smooth. A weight of one pound is fastened to the outer end of the cord. Find the position of equilibrium.

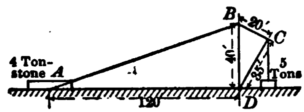
10. A and B are two fixed pulleys, 20 inches apart. A looped string 60 inches long passes over them. To one part of the loop is fastened a weight of 5 pounds. On the other part is a



movable pulley with a weight of 10 pounds. Find the position of equilibrium, assuming AB horizontal.

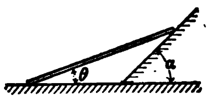
11. A weight of 100 pounds rests on a horizontal plane. $f = \frac{1}{6}$. A cord making an angle of 30° with the horizontal pulls on the weight. What is the tension in the cord when the body is about to move?

12. Given a derrick with conditions as shown in figure. (1) Which will move, the weight to be lifted or the stone anchor? Assume $f = 0.6$.



(2) Assume A fixed and each member of the derrick a two-force piece. Find the stress in each member due to the load of 5 tons.

13. In the figure shown, consider the coefficient of friction between the plane and the bar to be f', and that between the floor and the bar to be f. Let the length of the bar $= 2a$, the angle which the bar makes with the floor equal θ, and the angle which the plane makes with the floor equal α. Find the position of equilibrium.



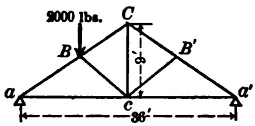
14. A plank 10 feet long leans against a carpenter's bench, which is $3\frac{1}{2}$ feet high. f in all cases is $\frac{1}{4}$. The angle which the plank makes with the vertical is 15°. Will the plank stand, or slip?

15. An elastic string, 18 inches long and in loop form, passes over two smooth pegs six inches apart, and at the same level. If 100 pounds would stretch the string to twice its natural length and if a smooth ring weighing 2 pounds is hung on the string, find the depth below the pegs at which the ring would come to rest.

16. A bar weighing 300 pounds rests with its ends on two smooth inclined planes, whose angles of inclination with the horizontal are 30° and 45°. Find the pressures on the planes.

17. Two weights of 5 pounds and 10 pounds respectively are fastened at opposite ends of a weightless rod 3 feet long. The rod with the weights is placed in a smooth spherical bowl 5 feet in diameter. Find the position of equilibrium of the rod.

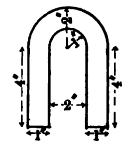
18. A tripod with legs 10 feet long supports a load of 1000 pounds. If the ends of the legs are at the vertices of an equilateral triangle, what is the length of one side of this triangle when the stress in each leg is 500 pounds?



19. Find the stresses in each member of the truss due to the load shown. Points B and B' are midway between a and C, and C and a' respectively.

20. Find the center of gravity of the area between a quadrant of a circle and a square, two of whose sides are the bounding radii of the quadrant.

21. Find the center of gravity of a spherical sector.



22. Find the center of gravity of the U-section shown.

23. Find the center of gravity of the frustum of a right circular cone, the radii of whose bases are 2 inches and 6 inches respectively, and whose altitude is 12 inches. The heaviness of the cone from which the frustum was obtained varied directly as the distance from the vertex, being 160 pounds per cubic foot at the vertex and 320 pounds at the base.

24. The steel in the tension side of a reënforced concrete beam consists of ten bars arranged in three rows. In the upper row there are two $\frac{3}{4}''$ square bars, in the middle row; four $\frac{3}{4}''$ square bars, in the bottom row, four $\frac{7}{8}''$ square bars. If the top row is 22 inches from the upper side of the beam, and the rows are two inches apart vertically, center to center of bars, what is the distance from the upper side of the beam to the center of gravity of the steel?

25. Find the center of gravity of a hemisphere whose heaviness varies as the distance from its geometric center.

26. Find the center of gravity of the trapezoid $ABCD$, AB and CD being the parallel sides: (a) using the calculus, (b) by dividing into finite parts, (c) graphically.

27. Find the moment of inertia of the frustum of a cone described in exercise 23: (a) about its geometric axis, (b) about a diameter of the larger base.

28. Find the moment of inertia of the hemisphere described in exercise 25 (page 282): (*a*) about a diameter of the great circle, (*b*) about a line perpendicular to the great circle at its center.

29. (*a*) Find the moment of inertia about the x-axis of that part of the rectangle ah which is outside the parabola,

$$\frac{y^2}{h^2} = \frac{x}{a}.$$

(*b*) Find the center of gravity of the same area.

30. Find the moment of inertia about its geometric axis of a flywheel whose rim is a ring 6 inches wide and 2 inches thick, its external diameter being 4 feet; the hub is a right circular cylinder 8 inches long and 8 inches in diameter; the eight spokes are cylinders running from hub to rim, whose cross sections are ellipses, the major axes being 3 inches, the minor axes 2 inches. The flywheel is made of cast iron which weighs 450 pounds per cubic foot.

31. From a right cylinder of cast iron, 4 inches long and 2 inches in diameter, has been removed a right cone whose base coincides with one end of the cylinder and whose altitude is 3 inches. This conical space is then filled with lead. Find the center of gravity of the solid thus formed. Cast iron weighs 0.26 pounds per cubic inch; lead, 0.41 pounds per cubic inch.

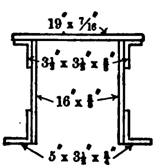
32. Find the moment of inertia of the frustum of a right circular cone, the radii of the upper and lower base being one and three inches respectively and the altitude being 10 inches: (*a*) about the geometrical axis of the frustum; (*b*) about a diameter of the smaller base.

33. A horse is hitched to a stone-boat by tugs which make an angle of 20° with the ground. If the pull in each of the tugs is 275 pounds, what force is the horse exerting parallel to the ground?

34. An I-beam 40 feet long and weighing 150 pounds per foot is supported from a horizontal deck by two ropes each 18 feet long attached to the ends of the I-beam and to points which are 50 feet apart. If each rope sustains one half the weight, what is the tension in each rope?

35. A trough, whose smooth sides make an angle of 90° with each other supports two smooth cylinders, one weighing 500 pounds, and the second weighing 125 pounds. The cylinders are of the same length; the larger one is in the corner of the trough. Find the pressure between the cylinders, and the pressure of each cylinder upon the trough, when the side of the trough upon which the smaller cylinder rests makes an angle of 30° with the horizontal.

36. A bar CD is pinned to a wall at C, and held in a horizontal position by a cable AB, which makes an angle of 40° with the bar. If CD is 16 feet and CB 10 feet, find the stress in AB and the pressure of the pin at C against the bar CD due to a load of 400 pounds at D.



37. Find the moment of inertia of a section made up of three plates and four angles as shown in figure.

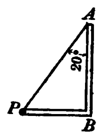
38. A uniform bar of wrought iron, which is 36 inches long and weighs 10 pounds, is supported by two pegs 20 inches apart, the middle of the bar being midway between the pegs. There is an unknown weight P attached to the bar at some point between the supports. It is found that if a weight of 20 pounds is suspended from one end of the bar, equilibrium is about to be broken. A weight of 15 pounds applied at the other end produces the same effect. What is the amount of P and at what point is it applied?

39. A block weighing 20 pounds is so held on a smooth plane which makes an angle of 30° with the horizontal that an elastic

string attached to the block and parallel to the plane is taut but under no stress. If a force of 1 pound causes the string to elongate 1 foot, and if the support is removed and the block is allowed to slide, how far will it go before coming to rest again? What is the time required for the motion and what is the greatest velocity of the block? Assume that the elastic limit of the string is not passed.

40. A man standing on the back platform of a train which is moving at the rate of 60 miles per hour throws a baseball back along the track. The ball leaves his hand moving in a horizontal direction with a velocity of 40 feet per second. If the hand is 10 feet above the track, with what velocity and at what distance from the point from which it was thrown, will the ball strike the track?

41. AP is a stiff weightless bar, which can move without friction about A in a vertical plane. A ball at P whose weight is W, rests against the horizontal bar PB. If the angle $PAB = 20°$, with what angular velocity ω must the entire mechanism revolve about the vertical axis AB, in order that the pressure of P against PB may be zero, assuming $AP = 18$ inches?



Find the amount $d\omega$ by which ω must be increased if AP be 18.1 inches long, in order that the horizontal pressure of P may be zero.

42. A door is supported by two hinges four feet apart. The upper hinge is so arranged that it can offer resistance to a horizontal motion only. What are the pressures on the hinges if the door weighs 15 pounds and the center of gravity of the door is $1\frac{1}{2}$ feet from the vertical through the hinges?

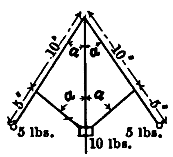
43. A weight of 10 pounds falls from a height of 6 feet upon a 40-pound spring. (A 40-pound spring is a spring which a force of forty pounds will compress one inch.) Assume that after the weight comes in contact with the spring it is held there, and trace the succeeding motion, determining the dis-

tance that the spring will be compressed, the velocity at any instant in the succeeding vibration, and the time of a vibration.

44. Would the time of a vibration be increased or diminished if the spring of exercise 43 were replaced by a 20-pound spring? Find the amount of the change.

45. Find the power required to haul a car weighing 60 tons up a grade which rises one foot in 100 feet, assuming that the friction is 8 pounds per ton and the velocity 25 miles per hour. How far would the car go if the power were shut off?

46. Find the horse-power of a steam pump that can raise 1500 gallons of water 100 feet in four minutes.



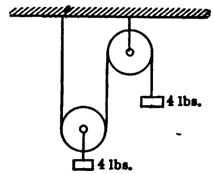
47. The governor in the figure rotates at such a speed that the angle α is 25°. Find the forces acting on the longer bars and the stress in the shorter ones. All connections are pin connections.

48. Through what distance must a force of 10 pounds act upon a body weighing 30 pounds to change its velocity from 15 feet per second to 20 feet per second? What time is required for the change of velocity?

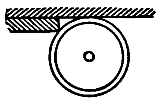
49. A flywheel is found to have 1,500,000 foot-pounds of energy when its speed is 200 R.P.M. An additional load reduces its speed to 150 R.P.M. in 2 seconds. What is the average horse-power produced by the slowing down?

50. An electric car weighs 10 tons. The driving wheels and axle weigh 1200 pounds, and the other axle and its wheels weigh 650 pounds. The radius of gyration of each axle with its wheels is 12 inches. The diameter of the wheels at the tread is 30 inches. What is the total kinetic energy of this car when traveling at the rate of 30 miles per hour? What portion of the energy is due to the rotation of the wheels and axles?

51. How long will it take the four-pound weight on the right to reach the floor, six feet below? Cord is flexible and inextensible. The pulleys are assumed weightless and frictionless.

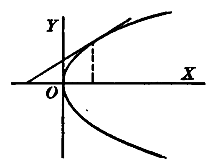


52. A set of locomotive drivers, six feet in diameter, are on a flat car of a train that has a velocity of 30 miles per hour. If the train stops with a uniform retardation in going 1000 feet, how high must the blocking be to keep the wheels from rolling?



53. A basket with a two-pound weight inside it has a cord passing over a frictionless pulley and a nine-pound weight fastened to the other end. If the system is left free to move, what is the pressure of the two-pound weight on the bottom of the basket? Neglect weight of cord and basket.

54. A man walks along a parabolic path, $y^2 = x$, at the rate of 4 miles per hour. What is his velocity parallel to the x-axis when his distance from the y-axis equals the focal distance?



55. A man walks at the rate of 4 miles per hour and rain is falling vertically with a velocity of 52 feet per second. At what angle must he carry a sheet of paper so as not to wet it?

56. An icy roof which has a slope of 45° is 20 feet from ridge to eaves. A particle of ice starts from the ridge, slides without friction down the roof, thence falling to the street 30 feet below. How far from the house does it strike the street?

57. A train has a velocity of 30 miles per hour. There is an abrupt drop of $\frac{1}{10}$ of an inch in the rail. Show that the truck wheels will strike the rail about 1 foot from the drop.

58. Suppose the earth gradually stopped rotating. Show that when it came to rest the top of Bunker Hill monument

would be $4\frac{1}{2}$ inches out of plumb. Lat. of monument 42° 22′; height, 220 feet.

59. Show that an express train running westward at the rate of 60 miles per hour weighs 300 pounds more than it would running eastward at the same rate. Weight of train 256 tons; latitude of train 42° 45′.

60. Show that the attraction between the earth and the sun is about $36\,(10)^{17}$ long tons.

If wires capable of sustaining the weight of a man weighing 180 pounds were to hold the earth in its orbit instead of gravitation doing it, how near together must such wires be placed?

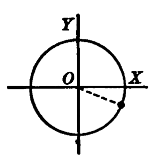
61. Suppose a closed can filled with a liquid were rotating with an angular velocity of ω. If the diameter of the can is $2\,a$ and the heaviness of the liquid is ρ, prove that the upward pressure on top of the can equals

$$\frac{\pi\omega^2 a^4\rho}{4}.$$

62. Suppose a two-inch stream of water from a fire engine is under a pressure great enough to cause it to rise 150 feet. How great a pressure would it exert on the panel of a door?

63. How long must a simple pendulum be in order to vibrate in the same time as a compound pendulum made by placing particles of equal weights at the vertices of a regular hexagon and suspending it from a vertex?

64. A man of weight W_1 walks on a turn table of weight W_2. The turn table is pivoted at its center and is free to turn without friction. Compare the angle θ through which he goes and the angle ϕ through which the turn table rotates.



65. A weight W_1 is fastened to the rim of a uniform circular disk of weight W_2. If the disk is in a vertical plane with the weight on a level with the center of the disk and if the weight is allowed to fall, how fast will the disk be turning when the weight reaches the lowest point?

66. A ball spinning vertically is dropped on a table. If ω is the angular velocity of the ball, how far will it slide and roll?

67. A billiard ball is placed in a spherical punch bowl, a line joining the center of the bowl and the ball making an angle θ with the vertical. The ball is then let go. Find the length of an equivalent simple pendulum.

68. A cylinder of weight W' rests on a rough horizontal table. A cord is wound about it, passes over a pulley at the edge of the table, and is attached to a weight W. The pulley is so arranged that the cord is horizontal. Find how long it will take the center of the cylinder to reach the edge of the table.

69. Water weighs 62.5 pounds per cubic foot. A cylindrical tube 2 feet long, with a cross section of 1 square inch is half filled with water. The tube is rotated in a horizontal plane at the rate of 300 revolutions per minute. Find the pressure on the outer end of the tube.

70. A right circular cylinder whose bases are horizontal and have a radius of 1 foot, and are separated 1 inch, is half filled with water. It is rotated about an axis perpendicular to the plane of the bases, at the rate of 300 revolutions per minute. Find the pressure per square inch on the lateral surface of the cylinder.

71. The speed of a railroad train was determined by observing the time which elapsed while covering the distance between successive telegraph poles. If the poles were 240 feet apart and the time which elapsed while the train was covering the first distance was 6 seconds, and the time while covering the second distance was 5 seconds, what was the acceleration of the train? What was the average velocity of the train during the second period?

72. A man weighing 160 pounds is riding a bicycle weighing 20 pounds up a slope that rises 4 feet in each 100 feet. His speed at the foot of the slope is 15 miles per hour; at the top of the

slope, which is 1000 feet long, his speed is 4 miles per hour. If the crank is $6\frac{1}{2}$ inches long, the diameter of the rear wheel 28 inches, and the gearing such that the rear wheel rotates 3 times while the crank rotates once, with what constant force must the man press upon the pedals that the change of speed may be uniform?

73. A solid cylinder, 18 inches in diameter and weighing 300 pounds, which can revolve freely on its axis without friction, has a cord wound around it and a weight of 20 pounds attached to the end of the cord. The support under the 20-pound weight is removed and the weight descends under the action of gravity. What is the kinetic energy stored in the cylinder when the unwinding cord has caused the cylinder to make three complete revolutions?

74. A thin circular plate 18 inches in diameter has two projecting axles 1 inch in diameter at its center. These axles rest on two parallel straight edges that make angles of 30° with the horizontal. The lower part of the plate projects downward between the straight edges. How long will it take the plate to roll 5 feet down the plane? Assume perfect rolling and neglect the weight of the axles.

75. How much difference should there be between the elevation of the inner and outer rails of a railroad track for a train moving with a speed of 45 M.P.H. around a curve of 2365 feet radius? The gauge is 4 feet $8\frac{1}{2}$ inches.

76. Two equal and perfectly elastic spheres are dropped at the same instant from different heights h and h' above a horizontal plane; determine whether their common center of gravity will ever rise to its original height.

77. A passenger noted that a lamp hanging freely in a coach of a train, which was running on a level track, made an angle of 3° with the vertical. What was the acceleration of the train in feet per second per second and miles per hour per hour?

78. Two weights of 10 and 15 pounds respectively are connected by a flexible inextensible string which passes over a

smooth frictionless pulley. The supports under the weights are suddenly removed. What is the tension in the string? How far will the 10-pound weight move in 3 seconds?

79. An electric car weighing 100 tons was moving down a grade of 2 feet in 100 at the rate of 60 miles per hour, when the emergency brakes were applied and the car stopped in 500 yards. What was the resistance offered by the brakes in pounds per ton? What time was required for the stopping, assuming that the retardation was uniform?

80. A flexible inextensible cable $ABCDE$ is hanging in a vertical plane, A and E being in the same horizontal line, and 15 feet apart. $AB = BC = CD = DE = 5$ feet. At B a weight of 10 pounds is suspended from a fixed pulley; at C, a weight of 12 pounds; and at D, one of 14 pounds is suspended. Find the angles that each portion of the cable makes with the horizontal, and the tension in each segment of the cable.

81. Two weights of 5 and 10 pounds each are attached to a circular disk at radii of 1 and 2 feet respectively. The angle between these radii is 120°. (*a*) Find the resultant force acting on an axle through its center when the disk is making 300 revolutions per minute and the 10-pound weight is vertically below the center. (*b*) Also find the amount and position of the weight which must be added to the disk at a distance of $1\frac{1}{2}$ feet from the center, that the force on the axle may be zero.

82. At a certain instant a motorboat is traveling due south at the rate of 20 miles per hour. A second motorboat which is 10 miles due east of the first boat is traveling southwest with a velocity of 30 miles per hour. If the boats continue to run in those directions with uniform speed, find how near the boats finally come to each other. What time has elapsed when the distance is shortest?

83. An elastic string, which stretches 1 inch under a pull of 5 pounds, is being used to raise a weight of 80 pounds, a

height of 6 feet. What work is required to raise the weight? If the work used in stretching the string is lost, what is the efficiency of this lifting device?

84. A weight of 20 pounds causes a spring to elongate 0.5 inches. If the spring with the weight attached is pulled down another inch and then released, find the time of vibration.

85. A 300-ton train moves down a 0.6 grade with a uniform velocity, the brakes not set. Upon reaching the level it runs half a mile before coming to rest. If the resistance remains constant, find the velocity of the train on the slope.

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